

## LETTER TO THE EDITOR

### The level density of a special quartic oscillator

S Tomsovic

Department of Chemistry, BG-10, University of Washington, Seattle, WA 98195, USA

Received 3 January 1991

**Abstract.** A refined asymptotic result is given for the integrated level density of the coupled quartic oscillators with potential  $V(q_1, q_2) = \alpha q_1^2 q_2^2$ . The method relies on an adiabatic separation in the heat kernel's spatial dependence out in the narrow channels of the equipotential surface.

Over the past few years, the system of coupled quartic oscillators with Hamiltonian

$$H = \frac{P^2}{2m} + \alpha q_1^2 q_2^2 \quad (1)$$

has received some attention both for its dynamical properties—it is nearly chaotic [11]—and its relevance to certain simplified field theories [2]. One of the interesting features is that its energy surface in phase space has four narrow, hyperbolic-shaped channels with infinite volume, aligned along either the  $q_1$  or  $q_2$  axes, yet the quantum version is proven to have a purely discrete spectrum [3]. The usual relationship between the density of states and the phase space volume cannot apply. Here we shall show how to refine Simon's [4] asymptotic result for the smoothed integrated level density.

The problem is quite like that encountered in the hyperbolic quantum billiards recently discussed by Steiner and Trillenber [5]. They were able to calculate improved asymptotic expansions for the level densities (or in the time domain, traces of the heat kernels) with the help of results by Van den Berg [6] for 'horn-shaped regions'. His results are applicable only to billiard problems, but the essential idea needed is the same. Namely, far out in the narrow channel regions there exists an adiabatic separation in the degrees of freedom along and perpendicular to the axes of the channels. This method adapts easily to the quantum version of (1) and, strictly speaking, requires some additional mathematical justification to be rigorous; we shall not give that here in order to allow intuitive presentation.

Consider the quantum Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + \alpha q_1^2 q_2^2 \quad (2)$$

whose classical analogue is given in the first equation. It is convenient here to allow  $(q_1, q_2)$  to serve a dual role, one as the canonical position variables in classical phase space and the other as the variables in the configuration space representation of the quantum system. The trace of its associated heat kernel is denoted  $Z(t)$  and as usual

$$Z(t) = \text{Tr}[\exp(-t\hat{H})] = \int_{-\infty}^{\infty} dq_1 dq_2 K(q_1, q_2; t) = \int_0^{\infty} dE e^{-tE} \rho(E) \quad (3)$$

where  $K(q_1, q_2; t)$  denotes the diagonal part of the kernel. The last form is included in order to indicate that  $Z(t)$  and the density of states,  $\rho(E)$ , form a Laplace transform pair. The standard, lowest-order semiclassical approximation, good for small  $t$ , is to substitute the classical  $H$  for the quantum  $\hat{H}$  and to replace the trace by an integral over the phase space volume normalized by  $(2\pi\hbar)^{-2}$ . The implication for  $K(q_1, q_2; t)$  is

$$K(q_1, q_2; t) \sim \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} dp \exp \left[ -t \left( \frac{p^2}{2m} + \alpha q_1^2 q_2^2 \right) \right] = \frac{m}{2\pi\hbar^2 t} \exp(-t\alpha q_1^2 q_2^2) \quad (4)$$

as long as neither  $|q_1|$  nor  $|q_2|$  becomes too large. More precisely, it is the products  $|q_1|t$  and  $|q_2|t$  which must remain small in the channels for (4) to be valid†. Therefore, it is convenient to break the  $(q_1, q_2)$  plane into two subdomains for integration. The first is a square centred at  $(0, 0)$  which excludes the narrow channels. In  $t \downarrow 0$ , its contribution to the trace,  $Z_q(t)$ , is found to be

$$\begin{aligned} Z_q(t) &= \int_{-q}^q dq_1 dq_2 K(q_1, q_2; t) \sim \frac{m}{2\pi\hbar^2 t} \int_{-q}^q dq_1 dq_2 \exp(-t\alpha q_1^2 q_2^2) \\ &\sim \left( \frac{m^2}{\pi\hbar^4 \alpha t^3} \right)^{1/2} \left[ \ln(2\sqrt{\alpha t} q^2) + \frac{\gamma}{2} \right] \end{aligned} \quad (5)$$

where  $\gamma = 0.5772 \dots$  is Euler's constant. The extent of the square,  $q$ , is understood to be as large as is consistent with the product  $qt$  tending to zero and only the leading order in  $q$  has been kept in the evaluation of the integral (which can be done in a straightforward fashion).

For the (diagonal part of) heat kernel in the large  $|q_1|$  channels, the adiabatic approximation is  $K(q_1, q_2; t) \sim K_{\text{free}}(q_1; t) * K_{q_1}(q_2; t)$  where  $K_{\text{free}}$  is the free particle kernel ( $\sim (m/2\pi\hbar^2 t)^{1/2}$ ) and  $K_{q_1}$  is the exact harmonic oscillator kernel whose only  $q_1$  dependence lies in the oscillator frequency (force constant). The four channels are identical and exponentially well isolated from each other, as can be deduced from the final form of (4). Therefore, we can cover the second subdomain by multiplying a single  $q_1$ -channel contribution by four and extending the  $q_2$  integration limits to  $\pm\infty$ . Then it is simplest to go after the expression for the integrated  $K_{q_1}$  directly because the trace of the harmonic oscillator kernel is a well known exponentiated spectral sum:

$$\begin{aligned} \int_{-\infty}^{\infty} dq_2 K_{q_1}(q_2; t) &= \sum_{n=0}^{\infty} \exp \left( -\frac{\hbar(2\alpha)^{1/2} q_1 t}{m^{1/2}} \left( n + \frac{1}{2} \right) \right) \\ &= \left[ 2 \sinh \left( \frac{\hbar\alpha^{1/2} q_1 t}{(2m)^{1/2}} \right) \right]^{-1}. \end{aligned} \quad (6)$$

The specific constants in the argument of the exponential are those appropriate for an oscillator frequency of  $\sqrt{2\alpha} q_1$ . In  $t \downarrow 0$ , the remaining contribution to the trace,  $Z_c(t)$ , is then

$$\begin{aligned} Z_c(t) &\sim \int_q^{\infty} dq_1 \frac{(2m)^{1/2}}{\hbar(\pi t)^{1/2} \sinh(\hbar\alpha^{1/2} q_1 t / (2m)^{1/2})} \\ &\sim - \left( \frac{m^2}{\pi\hbar^4 \alpha t^3} \right)^{1/2} \ln \left( \frac{\hbar^2 \alpha q^2 t^2}{8m} \right) \end{aligned} \quad (7)$$

† One way to view this restriction is based on the ket  $|q_1, q_2; t\rangle = \exp(-t\hat{H})|q_1, q_2; 0\rangle$ . The time must be short enough that the ket has not had the opportunity to 'discover' or reflect off the narrow width of the channel. See also the argument of the hyperbolic sine function of (6).

again to leading order in  $q$ . Summing the two contributions  $Z_c(t)$  and  $Z_q(t)$  gives

$$Z(t) = Z_c(t) + Z_q(t) \sim \frac{1}{8\sqrt{\pi\lambda t^3}} [-\ln(\lambda t^3) + \gamma + 4 \ln 2] \tag{8}$$

with  $\lambda = \hbar^4 \alpha / 16m^2$ . Due to the homogeneous nature of the potential, it was possible to know in advance that the various parameters for mass  $m$ ,  $\hbar$ , and strength of the interaction  $\alpha$  would combine to form just one scale constant and also that it would appear in the combination  $\lambda t^3$  as (8) confirms. In the same vein with this check is the disappearance of a  $q$  dependence which indicates a proper matching of the heat kernel approximations between the two subdomains. Some evidence suggests that the next non-vanishing contribution is  $O(t^{3/2} \ln t)$ †.

The inverse Laplace transform of  $Z(t)/t$  leads to an expression for the asymptotic smoothed integrated level density denoted by  $N(\mathcal{E}) (= \int d\mathcal{E} \rho(\mathcal{E}))$

$$N(\mathcal{E}) \sim \frac{1}{2\pi} \mathcal{E}^{3/2} \ln \mathcal{E} + \frac{\mathcal{E}^{3/2}}{3\pi} (2\gamma + 5 \ln 2 - 4) \tag{9}$$

where  $\mathcal{E} = E/\lambda^{1/3}$  is the unitless reduced energy. The log term was first given by Simon [4]. From a purely practical viewpoint, the log term by itself is of limited value since a scale change of  $\mathcal{E}$  introduces an arbitrarily important  $\mathcal{E}^{3/2}$  term. With the now derived  $\mathcal{E}^{3/2}$  term, the behaviour of the exact quantum integrated level density is perfectly followed right from the ground state; see figure 1. This method can be extended with little effort to one capable of generating the expansion to arbitrary order.

We gratefully acknowledge discussions with C Schmit, O Bohigas and D Ullmo. This work was supported in part by the National Science Foundation under grant no CHE-8507138. Computations were performed at the Institut de Physique Nucléaire, Orsay and the University of Washington.

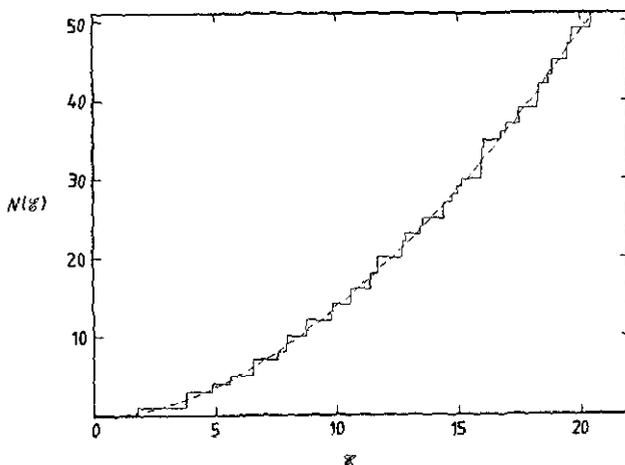


Figure 1. The comparison of the exact quantum integrated density of states (the 'staircase' function,  $N(\mathcal{E})$ ) to the asymptotic mean number expected for the lowest 50 eigenvalues.

† A preliminary calculation based on an extension suggests the next term is  $3\sqrt{\lambda t^3} \ln(\lambda t^3) / 20\sqrt{\pi}$ .

**References**

- [1] Dahlqvist P and Russberg G 1990 *Phys. Rev. Lett.* **65** 2837
- [2] Savvidy G K 1984 *Nucl. Phys. B* **246** 302
- [3] Simon B 1983 *Ann. Phys.* **146** 209
- [4] Simon B 1983 *J. Funct. Anal.* **53** 84
- [5] Steiner F and Trillenber P 1990 *J. Math. Phys.* **31** 1670
- [6] Van den Berg M 1984 *J. Funct. Anal.* **58** 150