Scattering approach to quantum electronic transport

Pier A. Mello
Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 México, Distrito Federal, Mexico

Steven Tomsovic
Department of Physics, FM-15, University of Washington, Seattle, Washington 98195
(Received 6 July 1992)

We study the problem of quantum electronic transport in a mesoscopic disordered system arranged in a two-probe configuration, using the transfer-matrix scattering formalism. We calculate the statistical average of quantities of physical interest starting from single-scattering units and working our way to the n-unit system by successive multiplication of the single-transfer matrices. In the weak-scattering limit discussed in the text, results are described by a diffusion equation in transfer-matrix space. Second moments of the departure of the single-transfer matrices from the unit matrix give rise to generalized diffusion coefficients: once these are specified, results are *universal,* in the sense that *higher-order moments are irrelevant.* The isotropy assumption (N-channel generalization of a random-phase assumption) made in some of our previous publications is not needed: we show that, as a consequence, the present results are physically more reasonable than those involving the isotropy assumption. We study the reflection and transmission amplitudes and coefficients for individual pairs of channels in the ballistic regime and the total-reflection coefficient in the ballistic and metallic regimes (the latter in the approximation of "equivalent channels"). The failure of isotropy in the localized regime is shown in one example. The present model, and not the one involving isotropy, is appropriate for the description of nonlocal quantum effects in multiprobe measurements, as well as persistent currents in normal metal rings.

I. INTRODUCTION

The scattering approach to quantum transport, initiated by Landauer,[1] has been applied by a number of authors to discuss the problem of disordered conductors. References 2–13 represent some of these contributions; for a review, the reader is referred to Ref. 14 and to the references contained therein.

In the approach of Ref. 8 a maximum-entropy criterion is invoked to find the statistical distribution of the transfer matrix. The result found is shown in Ref. 9 to arise from a *central limit theorem* (CLT) of a novel kind, thus representing a *universal limiting distribution,* once the overall inverse *mean free path* (mfp) (involving a summation over final channels and an average over initial ones) is specified; higher-order moments are irrelevant.

For a quasi-one-dimensional (quasi-1D) system employed in a two-probe configuration, the model of Refs. 8 and 9 gives excellent results, in agreement with perturbative calculations[15,16] and consistent with experiments.[17] The quantities that have been studied are (i) the average conductance that contains an Ohmic contribution and a quantum-mechanical weak-localization correction, (ii) universal conductance fluctuations, (iii) the backscattering enhancement, (iv) long-range correlations between pairs of transmission or reflection coefficients, and (v) time-reversal-symmetry breaking by a magnetic field.

Despite its successes, the earlier model that we are describing suffers from the rather unphysical "isotropy assumption," a generalization of the random-phase hypothesis in 1D, that was invoked for mathematical convenience; we shall thus refer to the model in question as the *isotropic model* (IM). As we shall explain in more detail in the next section, we can describe the scattering problem by an N-channel transfer matrix that can be written in a polar representation, as in Eq. (2.8) ahead. In the IM, the u,v,λ of Eq. (2.8) are statistically independent of one another and the unitary matrices u,v are distributed according to the invariant measure[18] of the unitary group $U(N)$. Intuitively, we can say that the N channels are mixed randomly among themselves, independently of the length L of the system: it is no surprise then that the IM does not allow the description of diffusion in the transverse direction. A related drawback of the IM is that the system dimensionality enters only through the number of channels [Eq. (2.1) below], which is correct only for quasi-1D systems. Another direct consequence of isotropy is that the backscattering enhancement mentioned in (iii) above is exactly 2 when the beam is reflected back to the incidence channel and 1 for reflection to other channels; this is in contrast to the cone seen experimentally and in other theoretical treatments. A similar comment applies for the correlations mentioned in (iv) above.

In a previous publication,[13] a more general approach, to be called the *nonisotropic model* (NIM), not needing the isotropy assumption was outlined. It is the purpose of this paper to give a more complete presentation of the method and extend some of the results presented there.

Isotropy was really the natural statistical assumption in the "polar representation" (2.8). In a more general analysis not using isotropy we have to learn how to han-

dle the problem in the "Cartesian representation," using the transfer-matrix elements themselves as our stochastic variables, fulfilling though the appropriate restrictions imposed by the requirements of flux conservation (FC) and time-reversal invariance (T symmetry).

In Sec. II we summarize the concepts behind the use of the transfer matrix in a scattering problem. The statistical problem is defined in Sec. III. The derivation of a multivariable diffusion equation can be found in the literature on random processes.[19] Nonetheless, in Appendix A we give the derivation for our particular problem, for the sake of completeness, and to make sure that we understand under what conditions it holds. Equations (3.18) and (3.21) [the latter in conjunction with (3.4)] represent the central results of the paper: they describe the evolution, with the length L of the system, of the expectation value of an observable related to the scattering problem. The result is *universal*, once the second moments that define the diffusion constants are specified. Appendix B gives some applications, including the standard CLT, of the general results found in Appendix A.

The remaining sections deal with applications of the central result mentioned above. Section IV studies the behavior of individual transmission and reflection amplitudes, while the total reflection coefficient is discussed in Sec. V. This latter quantity is a particular case of objects generally called traces, defined in Appendix C, where a simple example is worked out in detail. Appendix D gives an important relation, for averages of traces, between the IM and the present NIM. Appendixes E–H are aids to Appendix D. The polar parameters of Eq. (2.8) behave differently in the NIM than in the IM: an example of that different behavior is analyzed in Sec. VI. Our conclusions and some perspectives for future developments are presented in Sec. VII.

II. THE SCATTERING APPROACH: THE TRANSFER MATRIX

The scattering approach to be presented in this paper is based upon the notion of the *transfer matrix*, which has been studied in detail in various publications.[8,14,20] In this section we give the basic ideas behind this fundamental object and summarize its mathematical properties.

As indicated in the Introduction, we shall be concerned in this work with a two-probe configuration, in which the two reservoirs act as current source and sink and as voltage probes. We imagine the disordered system of interest (a wire of length L and width W) to be connected to the two reservoirs by means of perfect leads of the same width W. We introduce an x axis along the length of the system and $(d-1)$-dimensional y space in the transverse direction, d being the system dimensionality.

We deal with a confined geometry, so that boundary conditions at the lateral surface define, in the perfect leads,

$$N \sim (k_F W)^{d-1} \tag{2.1}$$

channels (or transverse modes), associated with traveling waves with Fermi momentum $\hbar k_F$ and Fermi energy $\epsilon_F = \hbar^2 k_F^2/2m$.

We assume our system to be at 0 K, so that inelastic effects are neglected altogether and our attention is focused on quantum-mechanical interference effects of waves with wave number k_F incident upon our system.

The wave function on either side of the scattering system is specified by a $2N$ component vector: the first N components are the amplitudes of the waves propagating to the right and the remaining N components are the amplitudes of the waves propagating to the left. The *transfer matrix* M relates the $2N$ components on the right of the system to those on the left. Knowing M, we can answer all questions related to the scattering process. For instance, we can find all the transmission and reflection coefficients between any pair of channels, using Eqs. (2.11) below.

The transfer matrix is closely related to a perhaps more familiar object: the scattering or S matrix, that relates incoming to outgoing waves.[14] However, the motivation for using M instead of S is its *multiplicativity property:* if we describe the disordered system as a sequence of "scattering units" (thin slices), with transfer matrices $M_1, M_2, \ldots, M_n$, respectively (assuming that between two successive units there is an interaction-free region, no matter how thin, where the wave function can be specified by the $2N$ vector introduced above[21]), the transfer matrix of the combined sequence is

$$M^{(n)} = M_n \cdots M_2 M_1 \ . \tag{2.2}$$

We require our problem to fulfill the properties of *flux conservation* (FC) and, in the absence of a magnetic field, *time-reversal invariance* (T symmetry). These two properties imply, for the transfer matrix, the relations

$$M \Sigma_z M^\dagger = \Sigma_z \tag{2.3}$$

and

$$M^* = \Sigma_x M \Sigma_x \ , \tag{2.4}$$

respectively. Here,

$$\Sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.5}$$

have the structure of Pauli matrices, 1 and 0 denoting the $N \times N$ unit and zero matrices, respectively. If we write the transfer matrix M in the form

$$M = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} , \tag{2.6}$$

T symmetry, Eq. (2.4), has the consequence that[8]

$$\gamma = \beta^*, \quad \delta = \alpha^* \ . \tag{2.7}$$

FC and T symmetry imply that the number of independent parameters of a transfer matrix is $N(2N+1)$. This fact can be taken care of explicitly by writing M in a *polar representation as*[8,20]

$$M = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} (1+\lambda)^{1/2} & \lambda^{1/2} \\ \lambda^{1/2} & (1+\lambda)^{1/2} \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix} , \tag{2.8}$$

where u, v are arbitrary $N \times N$ unitary matrices (each con-

tributing N^2 parameters) and λ is a real, diagonal matrix with N arbitrary positive elements $\lambda_1,\ldots,\lambda_N$.

Sometimes we shall find it advantageous to use the representation of Pereyra,[22]

$$M=\begin{bmatrix} e^{\theta} & 0 \\ 0 & e^{\theta^*} \end{bmatrix}\begin{bmatrix} \sqrt{1+\eta\eta^*} & \eta \\ \eta^* & \sqrt{1+\eta^*\eta} \end{bmatrix}, \tag{2.9a}$$

$$\theta=ih\ , \tag{2.9b}$$

where h is an arbitrary $N\times N$ Hermitian matrix (contributing N^2 parameters) and η is an arbitrary $N\times N$ complex symmetric matrix (contributing N^2+N parameters). The matrices θ,η can be expressed in terms of the matrices u,v,λ of Eq. (2.8) as

$$e^{\theta}=uv\ , \tag{2.10a}$$

$$\eta=v^{\dagger}\sqrt{\lambda}v^*\ . \tag{2.10b}$$

Quantities related to the scattering produced by a system will be called *observables;* they can all be expressed in terms of the transfer matrix M. Indeed, the $N\times N$ reflection and transmission matrices r,t (when incidence is from the left) and r',t' (when incidence is from the right) can be related to the elements $\alpha,\beta,\gamma,\delta$ of Eq. (2.6) as

$$\alpha=(t^{\dagger})^{-1},\quad \beta=r'(t')^{-1}\ , \tag{2.11a,b}$$

$$\gamma=-(t')^{-1}r,\quad \delta=(t')^{-1}\ . \tag{2.11c,d}$$

Equations (2.11) can in turn be expressed in terms of u,v,λ of Eq. (2.8), or h,η of Eq. (2.9).

An observable of particular importance is the conductance of our system measured in the two-probe configuration described above. In the theory of Ref. 5 it can be written as

$$g=2T\ , \tag{2.12}$$

in units of e^2/h; 2 is a spin factor and T is related to the transmission matrix t as

$$T=\mathrm{tr}tt^{\dagger}\ . \tag{2.13}$$

The statistical model and the related statistical properties of observables will be analyzed in the next section, using the notion of the transfer matrix that was introduced above.

III. THE STATISTICAL PROBLEM. THE EVOLUTION OF THE EXPECTATION VALUE OF OBSERVABLES

In what follows we consider not just one system, but a collection (ensemble) of systems having similar gross properties but differing in their microscopic configuration. Mathematically, we specify the problem by proposing a statistical model for the transfer matrices M_i of the slices; the goal is then to find the statistical properties, across the ensemble, of observables $F(M)$, M being the transfer matrix for the full system.

We formulate the problem in the following way. The basic composition law Eq. (2.2) can be written as the recursion relation

$$M^{(i+1)}=M_{i+1}M^{(i)}\ . \tag{3.1}$$

Later on we shall assume each unit to be a weak scatterer, so that the individual transfer matrices are close to the unit matrix, i.e.,

$$M_{i+1}=I+\varepsilon_{i+1}\ . \tag{3.2}$$

We then write Eq. (3.1) as

$$M^{(i+1)}=M^{(i)}+\delta M^{(i)}\ , \tag{3.3a}$$

with

$$\delta M^{(i)}=\varepsilon_{i+1}M^{(i)}\ . \tag{3.3b}$$

The multiplicative recursion relation (3.1) has been written as an additive relation in Eqs. (3.3), the added part $\delta M^{(i)}$ being now $M^{(i)}$ dependent. From the change (3.3) in the transfer matrix when one unit is added, one can calculate the change in the function $F(M)$: starting from its "initial" value $F(M^{(0)})$ one constructs F at "later" points by applying the recursion relation (3.3) step by step. We then take an ensemble average, assuming that the individual M_i's [and hence the ε_i's of Eq. (3.2)] are *statistically independent* of one another and *identically distributed* (therefore, the index i, that runs over the scattering units, will be dropped in most of the discussion ahead); the resulting equation then gives the "evolution" with the length of the system of expectation values of observables.

The program outlined above is carried out in detail in Appendix A, where the problem is formulated in a more general language, in order to develop a better insight. The application to the transfer-matrix problem that interests us in this section is then just a particular case. The results of Appendix A can be related to those found in studies of multidimensional random processes.[19] Their derivation, which is extremely simple, is presented in Appendix A for the sake of completeness, to make sure that we understand under what conditions they hold and to illustrate how the familiar CLT of statistics arises as a particular case [see Appendix B, part (1)].

We now apply to the present problem the general results of Appendix A (see also Appendix B, part (2), for a simpler but closely related application). Any observable $F(M)$ can be expressed in terms of the real and imaginary parts of all the matrix elements M_{ab}^{jk}, where $j=1,2$ and $k=1,2$ label the four $N\times N$ blocks in Eq. (2.6) and $a,b=1,\ldots,N$ label rows and columns in each block. Alternatively, F can be expressed in terms of all the complex quantities M_{ab}^{jk} and their complex conjugates; however, because of the relations (2.7), F can be written in terms of all the M_{ab}^{jk} alone. Therefore, the individual $(M_i)_{ab}^{jk}$ play the role of the variables $(\xi_i)_{\mu}$ of Appendix A and the $(M^{(n)})_{ab}^{jk}$ that of the "cumulative variables" $\xi_{\mu}^{(n)}$.

We mention in Appendix A that we allow for the possibility of there being a number of algebraic relations among our basic variables [see also Appendix B, case (3), for a simple illustration]. In the present case we indeed have the relations (2.3) and (2.4) among the M-matrix ele-

ments. Had we decided to work with the independent parameters of the "polar representation" (2.8), there would be no such algebraic relations; but then the recursion rule for those parameters would not be as easy to handle as that for the matrix elements themselves [Eqs. (3.3)] in the present "Cartesian representation."

The basic (exact) evolution equation (A18) for expectation values can be written, in the present context, as

$$\frac{\partial\langle F(M)\rangle_L}{\partial L}=\langle H(M)F(M)\rangle_L \ . \tag{3.4}$$

Here,

$$L=n\delta x \tag{3.5}$$

is the length of the system, consisting of n units spaced a distance δx, their density then being $\nu=1/\delta x$. We use the notation $\langle G(M)\rangle_L$ to indicate the average of $G(M^{(n)})$ evaluated with the probability distribution for $M^{(n)}$. The operator H is defined in Eq. (A15) as $\nu\mathcal{H};\mathcal{H}$, in turn, is expressed in Eq. (A13) in terms of the averaged translation operator D of Eq. (A10), which, in our problem, is given by

$$D(M)=\langle(\delta\dot{M})\rangle_\varepsilon\frac{\partial}{\partial\dot{M}}+\frac{1}{2!}\langle(\delta\dot{M})(\delta\ddot{M})\rangle_\varepsilon\frac{\partial^2}{\partial\dot{M}\partial\ddot{M}}+\cdots \ . \tag{3.6}$$

Here, δM is given by (3.3b) and the average is over ε, i.e.,

$$\langle\delta M\rangle_\varepsilon=\langle\varepsilon\rangle M \ . \tag{3.7}$$

The dots in (3.6) indicate a contraction over *all* the indices. For instance,

$$\langle(\delta\dot{M})\rangle_\varepsilon\frac{\partial}{\partial\dot{M}}=\sum_{\substack{jk\\ab}}\langle(\delta M)^{jk}_{ab}\rangle_\varepsilon\frac{\partial}{\partial\dot{M}^{jk}_{ab}} \ . \tag{3.8}$$

Using (A13), (A15), and (3.6), we thus find, for the evolution operator $H(M)$ appearing in (3.4), the expression

$$H(M)=\nu(D-\tfrac{1}{2}D^2+\cdots)=\nu\left[\langle(\varepsilon\dot{M})\rangle_\varepsilon\frac{\partial}{\partial\dot{M}}+\tfrac{1}{2}\langle(\varepsilon\dot{M})(\varepsilon\ddot{M})\rangle_\varepsilon\frac{\partial^2}{\partial\dot{M}\partial\ddot{M}}+\cdots-\tfrac{1}{2}\langle(\varepsilon\dot{M})\rangle\langle(\varepsilon\ddot{M})\rangle_\varepsilon\frac{\partial^2}{\partial\dot{M}\partial\ddot{M}}+\cdots\right] . \tag{3.9}$$

We specify the statistical distribution of ε by giving the statistics of the θ,η parameters of the individual scatterers introduced in Eq. (2.9). In this paper we consider the model[23]

$$\langle\theta\rangle=\langle\eta\rangle=0 \tag{3.10}$$

for the first moments, and

$$\langle\theta_{ab}\theta_{cd}\rangle=\kappa^{11,11}_{ab,cd} \ , \tag{3.11a}$$

$$\langle\theta_{ab}\theta^*_{cd}\rangle=\kappa^{11,22}_{ab,cd} \ , \tag{3.11b}$$

$$\langle\eta_{ab}\eta_{cd}\rangle=\kappa^{12,12}_{ab,cd} \ , \tag{3.11c}$$

$$\langle\eta_{ab}\eta^*_{cd}\rangle=\kappa^{12,21}_{ab,cd} \ , \tag{3.11d}$$

$$\langle\theta_{ab}\eta_{cd}\rangle=\kappa^{11,12}_{ab,cd}=0 \ , \tag{3.11e}$$

$$\langle\theta_{ab}\eta^*_{cd}\rangle=\kappa^{11,21}_{ab,cd}=0 \ , \tag{3.11f}$$

$$-\langle\theta^2\rangle=\langle\eta\eta^*\rangle \tag{3.11g}$$

for the second moments.

All higher moments, as well as all derivatives, appear in the exact evolution equation (3.4), with H given as in (3.9). Just as in the examples shown in the Appendix, we obtain an enormous simplification if we consider a limiting case, to be called the "weak-scattering limit" (WSL). Suppose that *for a given length* L of our system, Eq. (3.5), we let the number n of units grow and δx decrease, i.e.,

$$n\to\infty, \quad \delta x\to 0 \ ; \tag{3.12}$$

at the same time, while $\nu=1/\delta x\to\infty$, we let the individual units become infinitely weak, so that

$$\lim_{\substack{\nu\to\infty\\ \kappa\to 0}}(\nu\kappa^{jk,lm}_{ab,cd})=\sigma^{jk,lm}_{ab,cd} \tag{3.13}$$

are *finite quantities* [see Eqs. (B9) and (B25) for a simple illustration], while ν times a moment higher than the second vanishes in the limit. In particular, from Eq. (2.9) we see that

$$\lim_{\text{WSL}}\nu\langle|\varepsilon^{12}_{ab}|^2\rangle=\lim\nu\langle|\eta_{ab}|^2\rangle=\sigma^{12,21}_{ab,ab}\equiv\sigma_{ab}\equiv\frac{1}{l_{ab}} \ . \tag{3.14a}$$

From Eq. (2.11), this is, for a weak scatterer, the average reflection coefficient per unit length for the transition $b\to a$, and thus represents the inverse mean free path (MFP) for that process. Similarly,

$$\lim_{\text{WSL}}\nu\langle|\varepsilon^{11}_{ab}|^2\rangle=\lim\nu\langle|\theta_{ab}|^2\rangle=\sigma^{11,22}_{ab,ab}\equiv\sigma'_{ab}=1/l'_{ab} \tag{3.14b}$$

is the inverse MFP for forward scattering from channel b to a. We also define the inverse MFP l_a for channel a as

$$l_a^{-1}=\sigma_a=\sum_b\sigma_{ab}=\sum_b\sigma'_{ab} \ , \tag{3.14c}$$

where we have used Eq. (3.11g). In the framework of the NIM that we are discussing, when σ_{ab},σ'_{ab} are arbitrary quantities we speak of *nonequivalent channels* (NEC); for the choice (D7) for σ_{ab} we speak of *equivalent channels* (EC): the resulting l_a's are all equal.

In the above limiting process, the Fermi wave number k_F and the width W of the system are kept fixed, and so is the number of channels N of Eq. (2.1). In the present model, the electron sees a random medium with zero correlation length and its behavior is described through N-coupled modes. The WSL is certainly an oversimplification, though it gives a very rewarding result, as we shall now discuss.

We go back to the expression for H given in Eq. (3.9); we have to find the WSL of the various terms occurring

there. For the first term we have

$$\lim_{\mathrm{WSL}} \nu\langle \varepsilon^{11}\rangle = \lim \nu[\langle\theta\rangle + \tfrac{1}{2}\langle\eta\eta^*+\theta^2\rangle + \mathcal{O}(\theta\eta\eta^*,\theta^3,\ldots)] = 0 , \quad (3.15)$$

where we have used (2.9), (3.10), and (3.11g). In addition we have

$$\lim \nu\langle \varepsilon^{12}\rangle = \lim \nu[\langle\eta\rangle + \langle\theta\eta\rangle + \mathcal{O}(\theta^2\eta)] = 0 , \quad (3.16)$$

where we have used (2.9), (3.10), and (3.11e). The first term in (3.9) thus vanishes in the limit.

The second term in (3.9) is finite in the limit and can be expressed in terms of the quantities (3.13), i.e.,

$$\lim_{\mathrm{WSL}} \nu\langle \varepsilon^{jk}_{ab}\varepsilon^{lm}_{cd}\rangle = \sigma^{jk,lm}_{ab,cd} . \quad (3.17)$$

All the remaining terms in (3.9) vanish in the limit. We thus find the diffusionlike equation

$$\frac{\partial\langle F\rangle_L}{\partial L} = \tfrac{1}{2}\sigma^{jk,lm}_{ab,cd}\left\langle M^{kn}_{be}M^{mp}_{df}\frac{\partial^2 F}{\partial M^{jn}_{ae}\partial M^{lp}_{cf}}\right\rangle_L , \quad (3.18)$$

where a summation over repeated indices is understood. Equation (3.18) is the central equation of the present paper. It depends only on the second moments of the individual-unit distribution, *higher moments being irrelevant.* We thus have a CLT: once the second moments of ε are specified, Eq. (3.17), the limiting equation (3.18) is *universal,* i.e., independent of other details of the microscopic statistics. The quantities σ's appearing in Eq. (3.18) play the role of generalized diffusion coefficients. Different values of the σ's will in general generate different results for the expectation value of observables.

A CLT for the multichannel problem was proved in Ref. 9 within the *isotropy assumption* described in the Introduction. While the average of the individual M's vanishes in the isotropic model (IM), Eq. (3.10) shows that $\langle M_i\rangle$ is close to unity in the present NIM, which is more satisfactory for weak scatterers.

The CLT of Ref. 9 gave a diffusion equation [see Eq. (5.2)] for the joint probability density of the parameters λ_a of the polar representation (2.8). For thin slices of thickness δL (which, in the WSL, contain infinitely many of the original scattering units) it was possible to show that the solution of the diffusion equation is a distribution of maximum entropy, once the average inverse MFP, summed over final channels and averaged over initial ones, is fixed. We find that result extremely interesting; it actually justifies the maximum-entropy approach of Ref. 8. It is a highly nontrivial generalization of the situation occurring with the elementary CLT of statistics, which is analyzed in Appendix A: there, the resulting statistical distribution is a Gaussian, which is precisely the one of maximum entropy for a given centroid and variance.

Whether the basic result of the present section—the diffusionlike equation (3.18)—can be related to a maximum-entropy distribution with the constraints (3.15)–(3.17) is not known to us at the present time. In our view, it would be extremely important to answer this question.

We go back to Eq. (3.18). As we remarked above, in that equation there appear the second moments (3.17), that we now express in a more explicit fashion. Equations (3.11e) and (3.11f) imply

$$\sigma^{11,12}_{ab,cd} = \lim_{\mathrm{WSL}} \nu\langle\theta_{ab}\eta_{cd}\rangle = 0 , \quad (3.19a)$$

$$\sigma^{11,21}_{ab,cd} = \lim_{\mathrm{WSL}} \nu\langle\theta_{ab}\eta^*_{cd}\rangle = 0 . \quad (3.19b)$$

In addition, we make the following choices:[23]

$$\sigma^{11,11}_{ab,cd} = \lim_{\mathrm{WSL}} \nu\langle\theta_{ab}\theta_{cd}\rangle = -\lim_{\mathrm{WSL}} \nu\langle h_{ab}h_{cd}\rangle = -\delta_{ad}\delta_{bc}\sigma'_{ab} , \quad (3.20a)$$

$$\sigma^{11,22}_{ab,cd} = \lim_{\mathrm{WSL}} \nu\langle\theta_{ab}\theta^*_{cd}\rangle = \lim_{\mathrm{WSL}} \nu\langle h_{ab}h^*_{cd}\rangle = \delta_{ac}\delta_{bd}\sigma'_{ab} , \quad (3.20b)$$

$$\sigma^{12,21}_{ab,cd} = \lim_{\mathrm{WSL}} \nu\langle\eta_{ab}\eta^*_{cd}\rangle = \frac{\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}}{1+\delta_{ab}}\sigma_{ab} , \quad (3.20c)$$

$$\sigma^{12,12}_{ab,cd} = \lim_{\mathrm{WSL}} \nu\langle\eta_{ab}\eta_{cd}\rangle = 0 , \quad (3.20d)$$

where $\sigma'_{ab}, \sigma_{ab}$ are the inverse MFP's of Eqs. (3.14). Expanding the contractions in Eq. (3.18) we obtain, explicitly,

$$HF = \sum_{\substack{abcd\\km}} \sigma'_{ab}\left[M^{1k}_{bc}M^{2m}_{bd}\frac{\partial^2 F}{\partial M^{1k}_{ac}\partial M^{2m}_{ad}} - \tfrac{1}{2}M^{1k}_{bc}M^{1m}_{ad}\frac{\partial^2 F}{\partial M^{1k}_{ac}\partial M^{1m}_{bd}} - \tfrac{1}{2}M^{2k}_{bc}M^{2m}_{ad}\frac{\partial^2 F}{\partial M^{2k}_{ac}\partial M^{2m}_{bd}}\right]$$
$$+ \sum_{\substack{abcd\\km}} \sigma_{ab}\left[M^{1k}_{bc}M^{2m}_{bd}\frac{\partial^2 F}{\partial M^{2k}_{ac}\partial M^{1m}_{ad}} + M^{1k}_{ac}M^{2m}_{bd}\frac{\partial^2 F}{\partial M^{2k}_{bc}\partial M^{1m}_{ad}}\right] - \sum_{\substack{acd\\km}} \sigma_{aa}M^{1k}_{ac}M^{2m}_{ad}\frac{\partial^2 F}{\partial M^{2k}_{ac}\partial M^{1m}_{ad}} . \quad (3.21)$$

Right above Eq. (3.4), we mentioned that the M matrices satisfy Eqs. (2.3) and (2.4), arising from FC and T symmetry, respectively. These relations are satisfied by the M matrices of the individual scatterers, as is clear from the discussion right below Eq. (3.9), so that they must be satisfied for a system of any length. That relation (2.7) is satisfied is obvious from our construction. We shall verify that FC is exactly fulfilled as well. We call Q the matrix that appears on the left-hand side of (2.3), i.e.,

$$Q = M\Sigma_z M^\dagger . \quad (3.22)$$

We can use Eq. (3.21) to obtain the evolution equation for $\langle Q\rangle$. We find

$$\frac{\partial}{\partial L}\langle Q^{11}_{ab}\rangle_L=\delta_{ab}\sum_c(\sigma'_{ac}-\sigma_{ac})\langle Q^{11}_{cc}\rangle-(1-\delta_{ab})\langle Q^{11}_{ab}\rangle\ , \tag{3.23a}$$

$$\frac{\partial}{\partial L}\langle Q^{12}_{ab}\rangle_L=-\sigma'_{ab}\langle Q^{12}_{ba}\rangle\ , \tag{3.23b}$$

which we have to solve with the initial conditions

$$\langle Q^{11}_{ab}\rangle_0=\delta_{ab}\ , \tag{3.24a}$$

$$\langle Q^{12}_{ab}\rangle_0=0\ . \tag{3.24b}$$

Clearly,

$$\langle Q^{11}_{ab}\rangle_L=\delta_{ab}\ , \tag{3.25a}$$

$$\langle Q^{12}_{ab}\rangle_L=0\ , \tag{3.25b}$$

satisfy Eqs. (3.23) and the initial conditions (3.24), so that they are the solution. With a similar analysis for $\langle Q^{21}\rangle,\langle Q^{22}\rangle$, we then find

$$\langle Q\rangle=\Sigma_z\ . \tag{3.26}$$

Similarly, we can show that the evolution equations for $\langle Q^{11}_{ab}Q^{11}_{cd}\rangle_L,\langle Q^{12}_{ab}(Q^{12}_{cd})^*\rangle$ mix linearly only quantities of this type, with the various values of the indices a,b,c,d: we then say that the above quantities form a *closed set*. We can show that those equations are satisfied identically by

$$\langle Q^{11}_{ab}Q^{11}_{cd}\rangle_L=\delta_{ab}\delta_{cd}\ , \tag{3.27a}$$

$$\langle Q^{12}_{ab}(Q^{12}_{cd})^*\rangle_L=0\ . \tag{3.27b}$$

Equations (3.27) also give the right initial conditions. In particular, we have [recalling that $Q^{11}=(Q^{11})^\dagger$]

$$\langle|Q^{11}_{ab}|^2\rangle_L=\delta_{ab}\ , \tag{3.28a}$$

$$\langle|Q^{12}_{ab}|^2\rangle=0\ . \tag{3.28b}$$

Using (3.25) we have

$$Q^{11}_{ab}=\delta_{ab}\ , \tag{3.29a}$$

$$Q^{12}_{ab}=0\ , \tag{3.29b}$$

and thus

$$Q=\Sigma_z\ , \tag{3.30}$$

with no fluctuations. FC is thus satisfied identically for a system of any length.

IV. AVERAGE OF REFLECTION AND TRANSMISSION AMPLITUDES AND COEFFICIENTS FOR A THIN SLAB

We discuss in this section the application of the evolution Eqs. (3.4) and (3.21) to the reflection and transmission amplitudes and coefficients for a thin slab.

We first take, as the quantity F of Eqs. (3.4) and (3.21), the reflection amplitude r'_{ab} which, according to Eqs. (2.11b) and (2.11d), can be written as

$$r'_{ab}=(\beta\delta^{-1})_{ab}\ . \tag{4.1}$$

Straightforward application of the operator H of (3.21) gives

$$\partial_L\langle r'_{ab}\rangle_L=-(\sigma'_{ab}+\sigma_a+\sigma_b)\langle r'_{ab}\rangle_L\ . \tag{4.2}$$

The solution of (4.2) with the initial condition

$$\langle r'_{ab}\rangle_0=0 \tag{4.3}$$

is

$$\langle r'_{ab}\rangle_L=0\ . \tag{4.4}$$

Similarly, we consider the reflection amplitude r_{ab} which we write, from Eqs. (2.11c) and (2.11d), as

$$r_{ab}=-(\delta^{-1}\gamma)_{ab}\ . \tag{4.5}$$

Application of H gives

$$Hr_{12}=H^2r_{12}=\cdots=0\ , \tag{4.6}$$

so that

$$\langle r_{12}\rangle_L=0\ . \tag{4.7}$$

Equations (4.4) and (4.7) are a consequence of the randomness of η for individual scatterers, Eqs. (3.10) and (3.20d), and coincide with the results of the IM.

The transmission amplitude t_{ab} is given by Eqs. (2.11a) and (2.7) as

$$t_{ab}=(\delta^{-1})_{ba}\ . \tag{4.8}$$

Acting with H on (4.8) allows us to write

$$\partial_L\langle t_{ab}\rangle_L=-\sigma_a\langle t_{ab}\rangle_L\ . \tag{4.9}$$

The solution of this equation, with the initial condition

$$\langle t_{ab}\rangle_0=\delta_{ab} \tag{4.10}$$

is

$$\langle t_{ab}\rangle_L=e^{-\sigma_aL}\delta_{ab}\ . \tag{4.11}$$

Consideration of the transmission amplitude t'_{ab} does not give anything new, because $t'_{ab}=t_{ba}$.

Equation (4.11) is more reasonable than the result of the IM, which gives $\langle t_{ab}\rangle_L=0$ for arbitrary lengths and for all pairs of channels, including $L=0$ and $a=b$. The present NIM, in contrast, gives a "randomization length" for t_{aa} of the order of the MFP l_a.

The next quantity to be considered is $r'_{ab}r'^*_{a'b'}$. Application of H to it gives

$$\begin{aligned}H(r'_{ab}r'^*_{a'b'})=&(\delta^{a'}_a\delta^{b'}_b+\delta^{b'}_a\delta^{a'}_b-\delta^b_a\delta^{a'}_b\delta^{b'}_{a'})\sigma_{ab}-(\sigma'_{ab}+\sigma_a+\sigma_b+\sigma'_{a'b'}+\sigma_{a'}+\sigma_{b'})(r'_{ab}r'^*_{a'b'})\\&+\delta^{a'}_a\sum_c\sigma'_{ac}(r'_{cb}r'^*_{cb'})+\delta^{b'}_b\sum_c\sigma'_{bc}(r'_{ac}r'^*_{a'c})+\delta^{b'}_a\sum_c\sigma'_{ac}(r'_{cb}r'^*_{a'c})+\delta^{a'}_b\sum_c\sigma'_{bc}(r'_{ac}r'^*_{cb'})\\&+\sum_{cd}\sigma_{cd}(r'_{ac}r'^*_{a'c})(r'_{db}r'^*_{db'})+\sum_{c\neq d}\sigma_{cd}(r'_{cb}r'^*_{a'c})(r'_{ad}r'^*_{db'})\ .\end{aligned} \tag{4.12}$$

Equation (4.12) shows that the action of H on $(r'_{ab}r'^{*}_{a'b'})$ produces a quadratic combination of quantities of the same type: the monomials

$$(r'_{ab}r'^{*}_{a'b'}),\quad (r'_{cd}r'^{*}_{c'd'})(r'_{ef}r'^{*}_{e'f'}),\ \ldots \tag{4.13}$$

are thus mixed *linearly* by an application of H and form a *closed* set, in the language introduced right after Eq. (3.26). In the example of Appendix C it is the y_a's that, in Eq. (C6), form a closed set; in that example we can actually solve for the averages $\langle y_a \rangle$, with the result (C10). In a similar vein, we would like to find the average of the monomials (4.13), from which moments of reflection coefficients could then be obtained. We have to leave for the future the development of such an ambitious program, and devote the rest of this section and the following one to the calculation of a few properties of the reflection and transmission coefficients in various regimes and approximations.

Before proceeding, however, let us mention that monomials like those of (4.13), but with r' replaced by r, do not form a closed set: application of H gives terms that do not reproduce quantities of the same type again. The reason for the different behavior of the latter monomials and those of (4.12) can be understood very clearly by combining two scatterers. Consider a scatterer that may represent a given piece of wire, described by the reflection and transmission matrices r,r',t,t'; we add to it, to the right, a second scatterer, described by r_1,r'_1,t_1,t'_1. The reflection matrix $r'_{\rm tot}$ of the total system when incidence is from the right is

$$r'_{\rm tot}=r'_1+t'_1r'\frac{1}{1-r_1r'}t_1\ , \tag{4.14}$$

and the reflection matrix $r_{\rm tot}$ associated with incidence from the left is

$$r_{\rm tot}=r+tr_1\frac{1}{1-r'r_1}t'\ . \tag{4.15}$$

In (4.14), the original scatterer appears only through r', while it appears through r,r',t,t' in (4.15).

We go back to Eq. (4.12) and set $a'=a,b'=b$ to obtain the slope of $\langle R'_{ab}\rangle_L$. For $L=0$ only the constant term survives, giving

$$\left[\frac{\partial\langle R'_{ab}\rangle_L}{\partial L}\right]_{L=0}=\sigma_{ab}\ . \tag{4.16}$$

Acting with H on (4.12) gives the curvature. At $L=0$ we find

$$\left[\frac{\partial^2\langle R'_{ab}\rangle_L}{\partial L^2}\right]_{L=0}=-2(\sigma_a+\sigma_b)\sigma_{ab}+\sum_c(\sigma_{ac}\sigma'_{cb}+\sigma'_{ac}\sigma_{cb})+2\left[\delta_{ab}\sum_c\sigma'_{ac}\sigma_{ca}-\sigma'_{ab}\sigma_{ab}\right]\ . \tag{4.17}$$

Averages and correlations of the R_{ab}'s and the T_{ab}'s were studied in Refs. 11 and 12 within the IM. It was found, for instance, that

$$\frac{\langle R_{aa}\rangle_L^{\rm IM}}{\langle R_{a\neq b}\rangle_L^{\rm IM}}=2\ , \tag{4.18}$$

as a consequence of isotropy. Result (4.18) is related to the coherent backward enhancement, as discussed in weak-localization theory. In the present NIM, the backward enhancement phenomenon develops as we increase L, as the δ_{ab} term already present in the curvature (4.17) indicates. The sharp drop of $\langle R_{ab}\rangle_L^{\rm IM}$ by a factor 2 as we go from $a=b$ to $a\neq b$, which is a reasonable approximation for quasi-1 D systems, should actually develop into a cone in a more realistic analysis (see Refs. 24 and 15, last paper), just as one sees in experiments.[25] However, we do not as yet have predictions from the present model beyond the ballistic regime.

It is instructive at this point to perform a purely classical calculation of the reflection coefficient $\langle R'_{ab}\rangle_L$ for a thin slab and make a detailed comparison with the above results. In the classical treatment we describe each unit by means of arbitrary transmission and reflection coefficients R'_{ab},T_{ab} and sum fluxes instead of amplitudes. In the WSL we find

$$\langle R'_{ab}\rangle_L^{\rm CL}=\sigma_{ab}L+\left[-2(\sigma_a+\sigma_b)\sigma_{ab}+\sum_c(\sigma_{ac}\sigma'_{cb}+\sigma'_{ac}\sigma_{cb})\right]\frac{L^2}{2}+\cdots\ . \tag{4.19}$$

We see that (4.16) and the first two terms of (4.17) are reproduced by the classical calculation. The square bracket in (4.17), however, is not reproduced, thus indicating its quantum-mechanical origin; indeed, the δ_{ab} term was already related to the quantum backward enhancement phenomenon. We thus conclude that quantum corrections in $\langle R'_{ab}\rangle_L$ are already visible in the ballistic regime, for a very thin slab ($L\ll l$), if we look at the curvature of $\langle R'_{ab}\rangle_L$.

In a vein similar to Eq. (4.12), we can now find the action of H on $t_{ab}t^{*}_{a'b'}$, with the result

$$H(t_{ab}t^{*}_{a'b'})=-(\sigma_a+\sigma_{a'})(t_{ab}t^{*}_{a'b'})+\delta^{a'}_{a}\sum_c\sigma'_{ac}(t_{cb}t^{*}_{cb'})+\sum_{cd}\sigma_{cd}(r'_{ca}r'^{*}_{ca'})(t_{db}t^{*}_{db'})+\sum_{c\neq d}\sigma_{cd}(r'_{ca}r'^{*}_{da'})(t_{db}t^{*}_{cb'})\ . \tag{4.20}$$

We notice that the monomials $t_{ab}t^{*}_{a'b'}$ in conjunction with the monomials (4.13) now form a closed set under the application of H. Setting $a=a',b=b'$ (4.20) gives the slope of $\langle T_{ab}\rangle_L$ which, calculated at $L=0$, allows us to write

$$\langle T_{aa}\rangle_L=1-\left[\sigma'_{aa}+2\sum_{b(\neq a)}\sigma'_{ab}\right]L+\cdots\ , \tag{4.21}$$

$$\langle T_{a\neq b}\rangle_L=\sigma'_{ab}L+\cdots\ . \tag{4.22}$$

The present NIM gives a more reasonable result than the IM, that gives $\langle T_{ab}\rangle_0 = 1/N$ for every pair of channels, including $a=b$.

V. THE AVERAGE OF THE TOTAL REFLECTION AND TRANSMISSION COEFFICIENTS

The total reflection and transmission coefficients

$$R = R' = \mathrm{tr}(r'r'^{\dagger}) , \tag{5.1a}$$

$$T = \mathrm{tr}(tt^{\dagger}) = N - R \tag{5.1b}$$

are particular cases of the *traces* defined at the beginning of Appendix C. Their importance, in the problem of quantum transport, stems from the fact that, in a two-probe configuration, the conductance is related to T as in Eq. (2.12).

As long as we can, we shall deal with the case of NEC [see right after Eq. (3.14c)]. The various equations simplify enormously, though, for EC. Appendix D shows that, *in the NIM with EC, the average of a trace and the distribution $w_L(\lambda)$ of the $\{\lambda_a\}$ are the same as in the IM.* The $w_L(\lambda)$, in turn, satisfies the diffusion equation[8,9]

$$\frac{\partial w_L(\lambda)}{\partial L} = \frac{2}{N+1} \times \sum_{a=1}^{N} \frac{\partial}{\partial \lambda_a}\left[\lambda_a(1+\lambda_a)J(\lambda)\frac{\partial}{\partial \lambda_a}\frac{w_L(\lambda)}{J(\lambda)}\right] , \tag{5.2a}$$

where

$$J(\lambda) = \prod_{a<b} |\lambda_a - \lambda_b| . \tag{5.2b}$$

Notice, however, that in the NIM the EC assumption is made at the level of the MFP, whereas the isotropy assumption of Refs. 8 and 9 involves all the phases as well, and is thus much more restrictive. Our assumptions in relation to a single scatterer, including phases, are given in Eqs. (3.10), (3.11), and (3.20) and are physically more acceptable than isotropy; for instance, instead of (3.15) and (3.16), the IM gives $\langle M\rangle_L = 0$ for any number of scatterers; also the average transmission amplitudes (4.11) and transmission coefficients (4.21) and (4.22) are more reasonable in the present NIM, even for EC, than in the IM. We conclude that applications of the IM to traces test the EC rather than the isotropy assumption and that, *for traces,* any improvement (provided by the present NIM) on the IM results must arise from NEC. A particularly simple illustration of the general result shown in Appendix D is presented in Appendix C, which studies the centroid of the λ spectrum.

We start our discussion of the average reflection coefficient $\langle R\rangle_L$ by considering a very thin slice in the *ballistic regime.*

First of all we have the initial condition

$$\langle R\rangle_0 = 0 . \tag{5.3}$$

The slope of $\langle R\rangle_L$ at $L=0$ is obtained by summing Eq. (4.16) over indices, with the result

$$\left[\frac{\partial\langle R\rangle_L}{\partial L}\right]_{L=0} = \sum_{ab}\sigma_{ab} = \sum_a \sigma_a = \sum_a \frac{1}{l_a} = \frac{N}{l} , \tag{5.4}$$

where we have used the notation of Eq. (3.14c). We have also defined the *overall MFP l*; with this definition, the slope of $\langle R\rangle_L$ at $L=0$ for NEC coincides with that for EC with MFP l, i.e.,

$$\left[\frac{\partial\langle R\rangle_L^{\mathrm{EC}}}{\partial L}\right]_{L=0} = \frac{N}{l} . \tag{5.5}$$

Similarly, Eq. (4.17) gives the curvature of $\langle R\rangle_L$ at $L=0$ as

$$\left[\frac{\partial^2\langle R\rangle_L}{\partial L^2}\right]_{L=0} = -2\sum_{abc}\sigma_{ab}\sigma_{bc} = -2\sum_a \sigma_a^2 = -2\sum_a \frac{1}{l_a^2} . \tag{5.6}$$

The curvature (5.6) is negative for arbitrary $\{\sigma_{ab}\}$. For a given overall MFP l, defined as in (5.4), the absolute value of the curvature is larger when the individual MFP's l_a have a spread around l. Equation (5.6) reduces to the IM result when we specialize to EC (i.e., $l_a = l$), in agreement with the general result of Appendix D, i.e.,

$$\left[\frac{\partial^2\langle R\rangle_L^{\mathrm{EC}}}{\partial L^2}\right]_{L=0} = -2\frac{N}{l^2} . \tag{5.7}$$

We can also calculate the third derivative of $\langle R\rangle_L$ at $L=0$, with the result

$$\left[\frac{\partial^3\langle R\rangle_L}{\partial L}\right]_{L=0} = 4\sum_{abcd}\sigma_{ab}\sigma_{bc}\sigma_{cd} + 4\sum_{abcd}\sigma_{ab}\sigma_{ac}\sigma_{ad} - 2\sum_{abcd}\sigma_{ab}\sigma'_{bc}\sigma_{cd} + 2\sum_{abc}\sigma_{ab}\sigma_{bc}\sigma_{ca} + 2\sum_{ab}\sigma_{ab}^3 - 2\sum_a \sigma_{aa}^3 . \tag{5.8}$$

For EC, Eq. (D7), this result reduces to

$$\left[\frac{\partial^3\langle R\rangle_L^{\mathrm{EC}}}{\partial L^3}\right]_{L=0} = 6\frac{N}{l^3} + 2\frac{N(N+3)}{(N+1)^2 l^3} , \tag{5.9}$$

precisely as one finds from the IM for arbitrary N.

Just as we did in the previous section, it is instructive to compare the above results with a classical calculation. Summing Eq. (4.19) over a,b and calculating, in addition, the next cubic term, we find

$$\langle R\rangle_L^{\mathrm{CL}} = L\sum_{ab}\sigma_{ab} - \frac{L^2}{2!}\sum_{abc}2\sigma_{ab}\sigma_{bc} + \frac{L^3}{3!}\sum_{abcd}[4\sigma_{ab}\sigma_{bc}\sigma_{cd} + 4\sigma_{ab}\sigma_{ac}\sigma_{ad} - 2\sigma_{ab}\sigma'_{bc}\sigma_{cd}] + \cdots . \tag{5.10}$$

We see that the classical calculation reproduces exactly the slope (5.4) and the curvature (5.6) of the quantum-mechanical NIM with NEC. It is interesting to notice that the bracket in (4.17), that has a quantum origin, cancels exactly upon summing over channels, leaving a purely classical result. Thus the first quantum correction in $\langle R \rangle_L$ appears in the third derivative. Indeed, only the first three terms of the third derivative (5.8) are reproduced by the classical calculation; these are the ones containing the largest number of channel summations: in the EC result (5.9) they give rise to the first term, corresponding to the highest power in an N^k expansion. The terms in (5.8) containing only three-, two-, and one-channel summations, that give rise to the second term in the EC result (5.9), are thus quantum-mechanical corrections. An incipient backward enhancement is now seen, for the first time, in the term containing three summations, that describes a process starting from one channel and going back to the same channel.

We thus conclude that quantum corrections are already visible in the ballistic regime, for a very thin slab ($L \ll l$), if we look at the third derivative of $\langle R \rangle_L$.

We can go considerably beyond the ballistic regime if we make the EC approximation, Eq. (D7), and assume that the number of channel N is very large. Using the equivalence of the NIM with EC and the IM for averages of traces that is shown in Appendix D, we can just take over the result found elsewhere[10,12] for the IM to give

$$\langle R \rangle_L^{\mathrm{EC}} = N\frac{L/l}{1+L/l} + \frac{1}{3}\left[\frac{L/l}{1+L/l}\right]^3 + \frac{1}{45N}\frac{15(L/l)^3+15(L/l)^4-21(L/l)^5-28(L/l)^6-8(L/l)^7-(L/l)^8}{(1+L/l)^7} + \cdots . \tag{5.11}$$

In Eq. (5.11) the length L of the system must satisfy

$$0 \leq L \ll Nl , \tag{5.12}$$

i.e., it must be small compared with the 1D localization length $\xi \sim Nl$. We emphasize here the comment made at the beginning of this section: *Eq. (5.11) is valid in the IM and in the NIM with EC; its derivation in the latter is more satisfactory, however, than in the IM, because it never needs the assumption of isotropy for individual scatterers.*

In the metallic regime,

$$l \ll L \ll Nl , \tag{5.13}$$

the first term in (5.11) gives a transmission coefficient $\langle T \rangle \sim Nl/L$, typical of a *diffusive or Ohmic behavior.* The second term in (5.11) was shown in Ref. 12 to describe a coherent *weak-localization correction,* of a quantum-mechanical origin. Since Eq. (5.11) is valid for L all the way down to $L=0$, it is instructive to expand it in powers of L and compare with the results found above for the ballistic regime. The slope (5.5), the curvature (5.7), and the first term of the third derivative (5.9) are reproduced by the expansion of the first term in (5.11), thus confirming its classical nature. In contrast, the $(L/l)^3$ powers arising from the second and higher terms in (5.11) generate the second term in (5.9) which, from the above discussion, indeed has a quantum-mechanical origin.

Still in the framework of the NIM with EC one can calculate the fluctuation of T, with the result[10,12]

$$\mathrm{var}T = \tfrac{2}{15} + \cdots , \tag{5.14}$$

valid again when $N \gg 1$. In Eq. (5.14) there is no N, L, or l dependence, so that these fluctuations are called *universal conductance fluctuations.*

The above results (5.11) and (5.14) are in remarkable agreement with microscopic Green's-function calculations for quasi-1D systems, indicating that for $L \gg W$ all channels are so thoroughly mixed that replacing all the l_a's by a common l is a good approximation.

The NEC generalization of Eqs. (5.11) and (5.14) is not known to us at the present time. Studies in that direction will be left for the future.

VI. STATISTICAL BEHAVIOR OF THE "ANGULAR AND RADIAL PARAMETERS" u_{ab}, λ_b

We have seen in the preceding sections that quantities other than traces are even more "sensitive" than traces to the relaxation of the isotropy assumption: they usually behave differently in the IM and in the NIM with EC [Eq. (D7), supplemented with some model for the σ'_{ab}], the latter performing better and the NECM being a further improvement.

In this section we turn to the "angular and radial parameters" (2.8) of our transfer matrices. In the IM, the distribution of the u,v is specified beforehand as the *invariant measure of* $U(N)$; the u,v,λ are also *statistically independent* among themselves. In the present model we have been able to calculate one property involving both the angular and the radial parameters, for a very long conductor in the localized regime. This represents the opposite extreme from the ballistic regime studied in detail in the previous sections.

Consider the quantity y_a defined in Eqs. (C6a) and (C2). In terms of the polar parameters it can be written as

$$y_a = \sum_b |u_{ab}|^2 \lambda_b . \tag{6.1}$$

For a *very long conductor in the localized regime,* the largest λ, λ_N, gives the main contribution to (6.1). For $\langle y_a \rangle_L, L \gg Nl$, we thus have

$$\langle y_a \rangle_L \approx \langle |u_{aN}|^2 \lambda_N \rangle_L . \tag{6.2}$$

On the other hand, we can write for $\langle y_a \rangle_L$, Eq. (C10), an expression similar to (C16) (i.e., in terms of the shortest eigen-MFP $\hat{l}_1$) as

$$\langle y_a \rangle_L \approx \tfrac{1}{2}\langle a|1\rangle (e^{2L/\hat{l}_1}-1)\langle 1|w\rangle , \tag{6.3}$$

where $\langle a|1\rangle$ denotes the ath component of the eigenvec-

tor $|1\rangle$ associated with $\hat{l}_1$. Equating (6.2) and (6.3), we thus have

$$\langle |u_{aN}|^2\lambda_N\rangle_L \approx \tfrac{1}{2}\langle a|1\rangle(e^{2L/\hat{l}_1}-1)\langle 1|w\rangle \ . \tag{6.4}$$

We thus conclude that the a dependence of $\langle |u_{aN}|^2\lambda_N\rangle$ occurs only through $\langle a|1\rangle$, i.e.,

$$\langle |u_{aN}|^2\lambda_N\rangle_L \propto \langle a|1\rangle \ . \tag{6.5}$$

As an example, consider a "separable" σ matrix, i.e.,

$$\sigma_{ab}=\tau_a\tau_b \ , \tag{6.6}$$

as it would arise in a 2D model with δ scatterers. Relation (3.14c) implies $\sigma'=\sigma$, so that, from (C7), $\hat{\sigma}=\sigma$. Diagonalization of the matrix $\hat{\sigma}$ now gives one eigenvector, $|1\rangle$, parallel to $|\tau\rangle$, i.e.,

$$|1\rangle=\frac{|\tau\rangle}{\langle\tau|\tau\rangle^{1/2}} \ , \tag{6.7}$$

with the eigenvalue

$$l_1^{-1}=\langle\tau|\tau\rangle \ , \tag{6.8}$$

and $N-1$ eigenvectors $|i\rangle$ perpendicular to $|\tau\rangle$,

$$|i\rangle\perp|\tau\rangle, \quad i=2,\ldots,N \ , \tag{6.9}$$

with the eigenvalue

$$l_i^{-1}=0, \quad i=2,\ldots,N \ .$$

Equation (6.5) then gives

$$\langle |u_{aN}|^2\lambda_N\rangle \propto \tau_a \ . \tag{6.10}$$

For the 2D δ-scatterer model mentioned above, one has

$$\tau_a \propto \frac{1}{\sqrt{N^2-a^2}}, \quad a=1,\ldots,N \tag{6.11}$$

so that

$$\langle |u_{aN}|^2\lambda_N\rangle_L \propto \frac{1}{\sqrt{N^2-a^2}} \ . \tag{6.12}$$

Equation (6.12) tells us that the worst transmitting eigenchannels (largest λ) are preferentially projected on the modes of large transverse kinetic energy. Such a behavior has indeed been found in numerical simulations[26] and is physically reasonable. This considerable anisotropy has to be contrasted with the result of the IM, in which $\langle |u_{aN}|^2\lambda_N\rangle$ is a independent.

It is conceivable that the behavior (6.12) changes as we go down in the λ spectrum; if this is the case, the present NIM gives rise to a statistical dependence between the u matrix and λ, which is absent in the IM.

VII. CONCLUSIONS

In this paper, the problem of quantum transport is formulated within the scattering approach. The present nonisotropic model (NIM) represents a more physical description than the one provided by the isotropic model (IM) of previous publications.[8–12] In the latter, the u,v,λ of the parametrization (2.8) are statistically independent of one another and u,v are distributed according to the invariant measure[18] of the unitary group $U(N)$ (the N-channel generalization of the random-phase model); these restrictive assumptions are not made in the present model.[21,23]

The central result of the paper, Eq. (3.18), describes the evolution of expectation values through a diffusion equation (in transfer-matrix space), in which the second moments (3.17) play the role of generalized diffusion coefficients: once these are specified, results are universal, in the sense that higher-order moments are irrelevant. Universality attains in the weak-scattering limit (WSL) only. In that limit, the electron sees a random medium with zero correlation length, while the Fermi wave number k_F, the width W of the system, and hence the number N of channels are kept fixed. If we do not take the WSL, we are left with a differential equation involving derivatives to all orders. What the influence is of the term occurring beyond the diffusive one is not known to us at the present time. Estimates for a number of quantities, using reasonable values for the parameters involved, seem to indicate a small correction. However, there may be quantities for which the zero correlation length of the random potential implied by the WSL is too severe an approximation. It would be interesting to study the nonzero correlation length case more closely.

Equation (3.21), in conjunction with (3.4), gives a more explicit expression for the evolution equation in terms of the various inverse MFP's σ_{ab},σ'_{ab}.

We must emphasize that flux conservation is fulfilled at all stages in the present calculation. In contrast, in Ref. 13 flux was conserved on average and then it was shown that the correction vanishes in the WSL. In the present paper we end up with the same evolution equation as in Ref. 13, but obtained in a more satisfactory way.

We have studied the average of individual reflection and transmission amplitudes (r'_{ab},t_{ab}) and coefficients (R'_{ab},T_{ab}) for a thin slab. Results are physically more reasonable than in the IM. It is shown that the average reflection coefficient $\langle R'_{ab}\rangle_L$ exhibits, in its curvature at $L=0$, an incipient backward enhancement of a quantum-mechanical origin. Results are compared with a classical calculation, in order to assess the physical origin of the various contributions.

We have also studied the total reflection and transmission coefficients. These are examples of *traces* that involve a summation of functions $f(\lambda_a)$, in the parametrization of Eq. (2.8). It is shown that in the NIM with equivalent channels (EC), in which the choice $\sigma_{ab}=(1+\delta_{ab})/(N+1)l$ is made, the average of a function of traces (and the statistical distribution of the $\{\lambda_a\}$ as well) is the same as in the IM; the evaluation of these quantities in the NIM with EC is, however, more satisfactory than in the IM, because it does not need the restrictive assumptions summarized in the first paragraph of this section. Thus successes of the IM for traces, like the ones discussed below, are to be interpreted as successes of the EC rather than of the isotropy assumption: for traces, any improvement on the IM must therefore arise from nonequivalent channels (NEC).

As indicated above, the reflection coefficient R is an example of a trace. Its average $\langle R\rangle$ is studied in the ballis-

tic regime for the NIM with NEC. In the diffusive regime, and for EC only, $\langle R \rangle$ and var$R$ are analyzed: they compare well with perturbative calculations performed for quasi-1D systems; although the results are the same as for the IM, the present ones are better founded, as discussed above.

In the EC calculation[10,12] mentioned in the preceding paragraph, advantage was taken of the fact that the monomials

$$T_k^n T_{k'}^{n'}, \ldots, k, k' = 1, 2, 3, \ldots, \tag{7.1a}$$

where

$$T_k = \sum_a \frac{1}{(1+\lambda_a)^k} \tag{7.1b}$$

form a "closed set," in the sense that the evolution of their average is a linear combination of them all; even if the set is infinite, one can set up an approximation scheme[10,12] for $N \gg 1$, and calculate $\langle R \rangle$, var R, etc., to various orders in $1/N$. For NEC, on the other hand, if we start out with R, successive applications of H, Eq. (3.21), generate "more detailed" quantities in terms of channel indices and we cannot identify a closed set: this can be seen if, in Eq. (4.12), we set $a' = a, b' = b$, and sum over indices. In contrast, the problem has a clearer structure if we start out with the monomials (4.13); indeed, as Eq. (4.12) makes clear, they form a closed set in the sense explained above. It seems to us that such a structure will have to be taken advantage of in future developments. It is important to learn how to treat the NEC case. One reason is that the EC results for $\langle R \rangle$, var$R$, etc., discussed above, depend on the system dimensionality only through the number of channels, Eq. (2.1). This is fine for quasi-1D system $L \gg W$, but when $L \sim W$ we expect a dimensionality dependence. For instance, the quantum logarithmic correction[27] to Ohm's law finds no room in the EC model. For NEC, the parameters σ_{ab} may depend on the channel indices which, in turn, are abbreviations for $(d-1)$ indices; there may thus be an explicit dimensionality dependence.

It is generally found that quantities other than traces are even more "sensitive" than traces to the relaxation of the isotropy assumption. It is observed, for instance, that the worst transmitting eigenchannels are preferentially projected on the modes of large transverse kinetic energy, as borne out by computer simulations[26] as well. This anisotropy and statistical dependence between u, v, λ of (2.8) are alien to the IM.

It has been shown elsewhere[28] that the present model, and not the IM, is appropriate for the description of *nonlocal quantum effects*[29–31] in multiprobe measurements. For instance, classically, a "dangling wire"[28,31] of length L_3 hanging from a wire of length L_{12} has no effect on the conductance of the latter, whereas quantum mechanically there is an effect, as there is in wave guides. For 1D conductors, and when the main wire is a perfect conductor and the hanging wire a disordered one with MFP l, the resulting averaged transmission coefficient is

$$\langle T_{12} \rangle = \tfrac{1}{2}[1 + e^{-3L_3/l}\cos(2k_F L_3)] , \tag{7.2}$$

showing clearly the nonlocal effect of the dangling wire. In addition, one finds that the variance of T_{12} vanishes for $L_3 = 0$ and attains the limiting value $\frac{1}{2}$ when $L_3 \to \infty$. Recently we have also found (and will be reported elsewhere[32]) that the present model and not the IM allows a description of a four-probe experiment: i.e., a π circuit in which two terminals are used to measure the current and the other two to measure a voltage difference. Just as in the "dangling-wire" problem, the IM kills very interesting interference effects, which are conspicuous, however, in the present formulation.

The application of the model discussed in this paper to the analysis of *persistent currents* will also be reported in a separate publication.[33]

ACKNOWLEDGMENTS

P.A.M. acknowledges support from the Consejo Nacional de Ciencia y Tecnología and from the Sistema Nacional de Investigadores; he also wishes to thank the Laboratorio de Cuernavaca of the Instituto de Física for its kind hospitality. He is also grateful to P. Pereyra for his useful comments on the WSL and on the average flux conservation, and for letting us know the parametrization (2.9) prior to publication. S.T. is supported by the National Science Foundation Grant No. CHE-90-14555. We also acknowledge the hospitality of E. Riedel and his National Science Foundation Grant No. DMR-91-20282. We are also grateful to J. L. Pichard, for letting us know his unpublished numerical results on anisotropy. We also acknowledge important comments of Y. Imry on the dangling-wire problem.

APPENDIX A: THE EVOLUTION OF EXPECTATION VALUES GENERAL FORMULATION OF THE PROBLEM

Consider the sequence ξ_i, $i = 1, 2, \ldots, n, \ldots$, of K-dimensional vector variables, with components $(\xi_i)_\mu$, $\mu = 1, \ldots, K$. From the "individual" ξ_i's we construct the "cumulative" quantity $\xi^{(n)}$ through the recursion relation

$$\xi^{(i+1)} = \xi^{(i)} + \delta\xi^{(i)} , \tag{A1a}$$

where the increment

$$\delta\xi^{(i)} = f(\xi^{(i)}, \xi_{i+1}) \tag{A1b}$$

depends on the "added" variable ξ_{i+1} and may depend on the "position" $\xi^{(i)}$ itself.

We allow for the possibility of there being a number of fixed algebraic relations among the K components $(\xi_i)_\mu (\mu = 1, \ldots, K)$, as well as among the K components $\xi_\mu^{(n)} (\mu = 1, \ldots, K)$. These relations must be i and n independent. This is the case with the examples (B27) and (B28) studied below as well as with the group properties, (2.3) and (2.4) in our scattering problem.

We now consider a function $F(\xi)$. Starting from its "initial" value $F(\xi^{(0)})$, we construct F at "later" points by applying the recursion relation (A1) step by step. In the first step we have

$$\xi^{(1)}=\xi^{(0)}+\delta\xi^{(0)} , \tag{A2a}$$

with

$$\delta\xi^{(0)}=f(\xi^{(0)},\xi_1) , \tag{A2b}$$

so that

$$F(\xi^{(1)})=\mathcal{D}_{\xi_1}(\xi^{(0)})F(\xi^{(0)}) , \tag{A3}$$

where the *translation operator* $\mathcal{D}_{\xi_1}(\xi)$ is defined through its Taylor expansion

$$\begin{aligned}\mathcal{D}_{\xi_1}(\xi)&=1+(\delta\xi_\mu)\frac{\partial}{\partial\xi_\mu}+\frac{1}{2!}(\delta\xi_\mu)(\delta\xi_\nu)\frac{\partial^2}{\partial\xi_\mu\partial\xi_\nu}+\cdots\\&=1+D_{\xi_1}(\xi) ,\end{aligned} \tag{A4a}$$

where a summation over repeated indices is understood; $\delta\xi$ is assumed to be small enough to insure the convergence of (A4a). From (A1b), $\delta\xi$ may depend on ξ; therefore, if we want to write $\mathcal{D}$ as an exponential, we have to agree that successive applications of $\partial/\partial\xi_\mu$ do not act on $\delta\xi_\mu$; alternatively, we can write

$$\mathcal{D}_{\xi_1}(\xi)=[e^{\delta\xi_\mu(\xi',\xi_1)\partial/\partial\xi_\mu}]_{\xi'=\xi} . \tag{A4b}$$

We can now iterate once more to get

$$\xi^{(2)}=\xi^{(1)}+\delta\xi^{(1)} , \tag{A5a}$$

with

$$\delta\xi^{(1)}=f(\xi^{(1)},\xi_2) , \tag{A5b}$$

f being the *same* function as in (A2b). F changes as

$$F(\xi^{(2)})=\mathcal{D}_{\xi_2}(\xi^{(1)})F(\xi^{(1)}) . \tag{A6}$$

The function of $\xi^{(1)}$ on the right-hand side can be written in terms of $\xi^{(0)}$ using the translation operator appearing in (A3), i.e.,

$$F(\xi^{(2)})=\mathcal{D}_{\xi_1}(\xi^{(0)})\mathcal{D}_{\xi_2}(\xi^{(0)})F(\xi^{(0)}) . \tag{A7}$$

Notice that, in (A7), $\mathcal{D}_{\xi_2}$ is applied first and $\mathcal{D}_{\xi_1}$ later; the reason for this is that the two operators act on the *same* variable $\xi^{(0)}$: this fact is familiar, for instance, in angular-momentum theory.

Proceeding this way n times we find

$$F(\xi^{(n)})=\mathcal{D}_{\xi_1}(\xi^{(0)})\cdots\mathcal{D}_{\xi_n}(\xi^{(0)})F(\xi^{(0)}) . \tag{A8}$$

We now define the statistical problem by assuming that ξ_i,ξ_j are *statistically independent* for $i\neq j$ and *identically distributed.* Averaging both sides of Eq. (A8) we find

$$\langle F(\xi)\rangle_n=\langle[\mathcal{D}(\xi)]^nF(\xi)\rangle_0 . \tag{A9}$$

We use the notation $\langle G(\xi)\rangle_n$ to indicate the average of $G(\xi^{(n)})$, evaluated with the probability distribution for $\xi^{(n)}$. In (A9), $\mathcal{D}(\xi)$ indicates the operator $\mathcal{D}_{\xi_1}(\xi)$ of (A4) averaged over ξ_1, i.e.,

$$\begin{aligned}\mathcal{D}(\xi)&=\langle[e^{\delta\xi_\mu(\xi',\xi_1)\partial/\partial\xi_\mu}]_{\xi'=\xi}\rangle_{\xi_1}\\&=1+\langle(\delta\xi)_\mu\rangle_{\xi_1}\frac{\partial}{\partial\xi_\mu}\\&\quad+\frac{1}{2!}\langle(\delta\xi_\mu)(\delta\xi_\nu)\rangle_{\xi_1}\frac{\partial}{\partial\xi_\mu\partial\xi_\nu}+\cdots\\&=1+D(\xi) .\end{aligned} \tag{A10}$$

The coefficients in (A10) may be ξ dependent, because of the ξ dependence of $\delta\xi$.

We can also write (A9) as

$$\langle F(\xi)\rangle_n=\langle e^{n\mathcal{H}(\xi)}F(\xi)\rangle_0 , \tag{A11}$$

where

$$\mathcal{H}(\xi)=\ln\mathcal{D}(\xi)=\ln[1+D(\xi)] \tag{A12}$$

is defined through its series expansion

$$\mathcal{H}=D-\tfrac{1}{2}D^2+\tfrac{1}{3}D^3-\cdots . \tag{A13}$$

D is defined in (A10), with $\delta\xi$ being assumed small enough to insure convergence of the various series.

To prepare the way for our applications to the scattering problem, imagine that we have a collection of scattering units spaced a distance δx (the density being $\nu=1/\delta x$ units per cm) and that we associate the variable ξ_i to the ith unit. We define

$$L=n\delta x , \tag{A14}$$

$$H=\nu\mathcal{H} , \tag{A15}$$

L being the length of our system; the rescaling introduced in (A15) will be useful later on.

Using (A14) and (A15), we write (A11) as

$$\langle F(\xi)\rangle_L=\langle e^{LH(\xi)}F(\xi)\rangle_0 . \tag{A16}$$

In principle, the *exact* result (A16) gives the explicit construction of the expectation value $\langle F\rangle_L$ for any length, starting with $L=0$. However, since the exponential operator may not be easy to handle, we convert (A16) into a differential equation by differentiating both sides with respect to L, i.e.,

$$\frac{\partial\langle F(\xi)\rangle}{\partial L}=\langle e^{LH(\xi)}[H(\xi)F(\xi)]\rangle_0 . \tag{A17}$$

Applying (A16) to the new function HF, we finally obtain

$$\frac{\partial\langle F(\xi)\rangle_L}{\partial L}=\langle H(\xi)F(\xi)\rangle_L . \tag{A18}$$

This is an exact equation that describes the evolution of the expectation value $\langle F\rangle_L$. It is applied in the text in Eq. (3.4). Some particular cases are studied in Appendix B.

APPENDIX B: APPLICATIONS OF THE GENERAL RESULT FOR THE EVOLUTION OF EXPECTATION VALUES

(1) We study the sum

$$\xi^{(n)}=\sum_{i=1}^{n}\xi_i , \tag{B1}$$

where ξ_i are c numbers, statistically independent and identically distributed. We can use the analysis of Appendix A, setting the number of components $K=1$ and writing the recursion relation (A1) as

$$\xi^{(n+1)}=\xi^{(n)}+\delta\xi^{(n)} , \tag{B2a}$$

$$\delta\xi^{(n)}=\xi_{n+1} . \tag{B2b}$$

The operators $\mathcal{D}$, Eq. (A10), and $\mathcal{H}$, Eq. (A13), are then given by

$$\mathcal{D}(\xi)=\langle e^{\xi_1(\partial/\partial\xi)}\rangle_{\xi_1} , \tag{B3a}$$

$$\mathcal{H}(\xi)=\ln\langle e^{\xi_1(\partial/\partial\xi)}\rangle_{\xi_1} , \tag{B3b}$$

which are recognized as the moment and cumulant generating functions of the variable ξ_1, respectively, i.e.,

$$\mathcal{D}(\xi)=\sum_{r=0}^{\infty}\frac{\mu_r}{r!}\frac{\partial^r}{\partial\xi^r} \tag{B4a}$$

$$\mathcal{H}(\xi)=\sum_{r=1}^{\infty}\frac{\kappa_r}{r!}\frac{\partial^r}{\partial\xi^r} , \tag{B4b}$$

μ_r,κ_r being the rth moment and cumulant of ξ_1, respectively.

As an example, suppose all the ξ_i are zero-centered Gaussian variables with variance κ_2. The only nonvanishing κ_r in (B4b) is thus κ_2, so that the evolution equation (A18) becomes

$$\frac{\partial\langle F(\xi)\rangle_L}{\partial L}=\frac{\nu\kappa_2}{2!}\left\langle\frac{\partial F(\xi)}{\partial\xi^2}\right\rangle_L . \tag{B5}$$

This is the correct result, because $\xi^{(n)}$ is a zero-centered Gaussian variable with variance $n\kappa_2$; its probability density satisfies a diffusion equation which, multiplied by $F(\xi)$, yields, after integration by parts, Eq. (B5), which will thus be called a "diffusionlike" equation.

If the ξ_i are not necessarily Gaussian variables, but are zero centered with variance κ_2, we have the general result

$$\frac{\partial\langle F(\xi)\rangle_L}{\partial L}=\nu\left\langle\frac{\kappa_2}{2!}\frac{\partial^2F(\xi)}{\partial\xi^2}+\frac{\kappa_3}{3!}\frac{\partial^3F(\xi)}{\partial\xi^3}+\cdots\right\rangle_L . \tag{B6}$$

This result is exact, but not particularly useful, since it involves an infinite number of derivatives. We obtain an enormous simplification if we consider a limiting case which, in the applications to the scattering problem, is called the "weak-scattering limit" (WSL), discussed in the text right after Eq. (3.11). We adopt the same momenclature here. Suppose that, *for a given length L* of our system, as given by (A14), we let the number n of scatterers grow and δx decrease, i.e.,

$$n\to\infty, \quad \delta x\to 0 , \tag{B7}$$

($\nu=1/\delta x\to\infty$); at the same time, we decrease the individual "intensity" κ_2

$$\kappa_2\to 0 , \tag{B8}$$

so that

$$\nu\kappa_2=\sigma \tag{B9}$$

is a *finite quantity*. Then, if the distribution of ξ_1 is such that, for $r>2$, $|\kappa_r|\sim\kappa_2^{r/2}$, we have

$$\nu|\kappa_r|\sim\nu\kappa_2^{r/2}=\sigma\frac{\kappa_2^{r/2}}{\kappa_2}=\sigma\kappa_2^{(r-2)/2} , \tag{B10}$$

which vanishes as $\kappa_2\to 0$, when $r>2$. Only the first term on the right-hand side of (B6) survives, giving

$$\frac{\partial\langle F(\xi)\rangle_L}{\partial L}=\frac{\sigma}{2}\left\langle\frac{\partial^2F(\xi)}{\partial\xi^2}\right\rangle_L . \tag{B11}$$

This is precisely the familiar result of the *central-limit theorem* (CLT), which asserts that, in the above limit, $\xi^{(n)}$ becomes a zero-centered Gaussian variable with variance

$$n\kappa_2=\sigma L . \tag{B12}$$

Only the first and second moments of the individual ξ_i are relevant for the final result. In this sense, the CLT gives a *universal* answer, once the first two moments are fixed.

As an elementary application of (B11), let us set $F(\xi)=\xi^2$. Then $\partial\langle\xi^2\rangle_L/\partial L=\sigma$ which, solved with the initial condition $\langle\xi^2\rangle_0=0$, gives $\langle\xi^2\rangle_L=\sigma L$, as in (B12).

(2) We study the product

$$t^{(n)}=t_n\cdots t_2t_1 , \tag{B13}$$

where t_i are positive definite numbers, statistically independent and identically distributed. We can apply all the results of the previous example if we define

$$\xi_i=\ln t_i . \tag{B14}$$

Alternatively, we can apply the general analysis with $K=1$ and the recursion relations (A1) written as

$$t^{(i+1)}=t^{(i)}+\delta t^{(i)} , \tag{B15a}$$

$$\delta t^{(i)}=\varepsilon_{i+1}t^{(i)} , \tag{B15b}$$

where we have written

$$t_i=1+\varepsilon_i , \tag{B16}$$

having in mind that the t_i's are close to unity.

The motivation for considering this example is that it has in common with the scattering problem the multiplicative nature of the combination law [compare Eq. (B13) with (2.2), and (B15b) with (3.3b)].

The operator $\mathcal{D}$ of (A10) is now given by

$$\mathcal{D}(t)=\langle[e^{\varepsilon_1t'(\partial/\partial t)}]_{t'=t}\rangle_{\varepsilon_1}=1+D(t)$$

$$=1+\langle\varepsilon_1\rangle t\frac{\partial}{\partial t}+\frac{\langle\varepsilon_1^2\rangle}{2}t^2\frac{\partial^2}{\partial t^2}+\cdots . \tag{B17}$$

Writing, in (B3a), $\xi_1=\ln(1+\varepsilon_1)$, $\partial/\partial\xi=t\partial/\partial t$, one can show the equivalence of (B4a) and (B17).

The operator $\mathcal{H}$ of (A13) now becomes

$$\mathcal{H}(t)=\langle\varepsilon_1\rangle t\frac{\partial}{\partial t}+\frac{\langle\varepsilon_1^2\rangle}{2!}t^2\frac{\partial^2}{\partial t^2}+\cdots$$
$$-\frac{\langle\varepsilon_1\rangle^2}{2}t\frac{\partial}{\partial t}t\frac{\partial}{\partial t}+\cdots+\cdots\ . \tag{B18}$$

Suppose that $\langle\varepsilon_1\rangle=0$ and, for a fixed length L given by (A14), we take the WSL as

$$n\to\infty\ , \tag{B19a}$$
$$\nu=(\delta x)^{-1}\to\infty\ , \tag{B19b}$$
$$\langle\varepsilon_1^2\rangle\to 0\ , \tag{B19c}$$

with

$$\nu\langle\varepsilon_1^2\rangle=\sigma \tag{B19d}$$

being a finite quantity. Then, the evolution equation (A18) becomes

$$\frac{\partial\langle F(t)\rangle_L}{\partial L}=\frac{\sigma}{2}\left\langle t^2\frac{\partial^2 F(t)}{\partial t^2}\right\rangle_L\ . \tag{B20}$$

Again, this is a universal result, in the sense that moments of ε_1 higher than the second are irrelevant.

As a simple application of (B20), set $F(t)=t$. Then $\partial\langle t\rangle_L/\partial L=0$, which tells us that $\langle t\rangle_L$ does not evolve with L. The initial condition $\langle t\rangle_0=1$ then gives $\langle t\rangle_L=1$, as we can verify directly from (B13), using $\langle\varepsilon_1\rangle=0$. As a second example, set $F(t)=t^2$. Then $\partial\langle t^2\rangle_L/\partial L=\sigma\langle t^2\rangle_L$ which, solved with the initial condition $\langle t^2\rangle_0=1$, gives $\langle t^2\rangle_L=e^{\sigma L}$, as we can again verify directly from (B13) using $\langle\varepsilon_1\rangle=0$ and the limit (B19).

(3) We generalize the sum (B1) to K-dimensional vectors

$$\xi^{(n)}=\sum_{i=1}^{n}\xi_i\ . \tag{B21}$$

The individual ξ_i,ξ_j are statistically independent if $i\neq j$; the individual distributions are identical, with

$$\langle(\xi_i)_\mu\rangle=0\ , \tag{B22}$$
$$\langle(\xi_i)_\mu(\xi_i)_\nu\rangle=\kappa_{\mu\nu}\ , \tag{B23}$$

but otherwise arbitrary. The operator $H(\xi)$ of (A15) and (A13) is then

$$H(\xi)=\nu\frac{\kappa_{\mu\nu}}{2}\frac{\partial^2}{\partial\xi_\mu\partial\xi_\nu}+\cdots\ . \tag{B24}$$

We now take the WSL as in the above examples, for a fixed L, Eq. (A14), and with

$$\nu\kappa_{\mu\nu}=\frac{1}{l_{\mu\nu}}\equiv\sigma_{\mu\nu}\ . \tag{B25}$$

We then end up with the evolution equation

$$\frac{\partial\langle F(\xi)\rangle_L}{\partial L}=\frac{\sigma_{\mu\nu}}{2}\left\langle\frac{\partial^2 F(\xi)}{\partial\xi_\mu\partial\xi_\nu}\right\rangle\ , \tag{B26}$$

that reduces to (B11) when $K=1$. Equation (B26) represents again a universal result once the $\sigma_{\mu\nu}$'s are specified, because it is independent of the higher-order moments of the individual distribution.

We use this case to illustrate the important point made right after Eq. (A1), i.e., that we may allow for certain algebraic relations among the K variables $(\xi_i)_\mu$ (i fixed) and among K the variables $\xi_\mu^{(n)}$ (n fixed). As an example, assume the relations

$$(\xi_i)_1=\cdots=(\xi_i)_K\ . \tag{B27}$$

Equation (B21) then implies

$$\xi_1^{(n)}=\cdots=\xi_K^{(n)}\ . \tag{B28}$$

In other words, the relations (B27) among the individual variables are "conserved", as we go along the chain, among the cumulative ones (B28); thus, the joint probability density of these K variables must involve the appropriate δ functions to enforce the above algebraic relations. Let us look at the description of this elementary problem in terms of the evolution equation (B26) of expectation values. Choose $F=(\xi_\mu-\xi_\nu)^2,\mu\neq\nu$. Equation (B26) then gives

$$\frac{\partial}{\partial L}\langle(\xi_\mu-\xi_\nu)^2\rangle_L=\sigma_{\mu\mu}+\sigma_{\nu\nu}-2\sigma_{\mu\nu}\ . \tag{B29}$$

The relations (B27) imply [from (B23) and (B25)]

$$\sigma_{\mu\mu}=\sigma_{\nu\nu}=\sigma_{\mu\nu}\ , \tag{B30}$$

so that

$$\frac{\partial}{\partial L}\langle(\xi_\mu-\xi_\nu)^2\rangle_L=0\ . \tag{B31}$$

If we start out our procedure with $\xi^{(0)}=0$, the initial condition is

$$\langle(\xi_\mu-\xi_\nu)^2\rangle_0=0\ . \tag{B32}$$

The solution of the evolution equation (B31) is then

$$\langle(\xi_\mu-\xi_\nu)^2\rangle_L=0\ , \tag{B33}$$

which implies

$$\xi_\mu=\xi_\nu\ , \tag{B34}$$

exactly as in (B28). We thus see that the conservation of the relation as we go along the chain is automatically described by our evolution equation of expectation values. In conclusion, not only may we allow for algebraic relation among our variables, but we can calculate the evolution of expectation values (even of quantities other than the F used above) without ever having to worry about the presence of δ functions in the actual integral that defines the expectation value. These observations are very important for our applications to the scattering problem.

APPENDIX C: THE CENTROID OF THE λ SPECTRUM

In the notation of Eqs. (2.6) and (2.8) we define the Hermitian $N\times N$ matrices

$$x=\alpha\alpha^\dagger=u(1+\lambda)u^\dagger\ , \tag{C1}$$
$$y=\beta\beta^\dagger=u\lambda u^\dagger=x-1\ . \tag{C2}$$

Quantities of the type

$$\mathrm{tr}f(y)=\sum_{a=1}^{N} f(\lambda_a) \tag{C3}$$

will be generally called *traces*. A particularly important example is the transmission coefficient T of Eq. (2.13), that can be written as

$$T=\mathrm{tr}(tt^{\dagger})=\mathrm{tr}(\alpha\alpha^{\dagger})^{-1}=\mathrm{tr}\frac{1}{x}=\mathrm{tr}\frac{1}{1+y}=\sum_{a=1}^{N}\frac{1}{1+\lambda_a} . \tag{C4}$$

The trace

$$\rho_k=\mathrm{tr}(y^k)=\sum_{a=1}^{N}\lambda_a^k \tag{C5}$$

defines the kth moment of the λ spectrum. We concentrate here on the centroid $k=1$. Applying the basic evolution operator H of Eq. (3.21) to the quantity

$$y_a=y_{aa} , \tag{C6a}$$

we obtain

$$Hy_a=2\sum_b \hat{\sigma}_{ab}y_b+\sigma_a , \tag{C6}$$

where σ_a was defined in (3.14c) and

$$\hat{\sigma}_{ab}=\tfrac{1}{2}(\sigma_{ab}+\sigma'_{ab}) \tag{C7}$$

is the arithmetic mean of the forward and backward inverse MFP matrices. We see that the quantities $y_a, a=1,\ldots,N$ transform linearly among themselves under the application of H: they form a *closed* set, in the terminology introduced right after Eq. (3.26).

Equation (C6) gives the evolution equation

$$\partial_L\langle y_a\rangle_L=2\sum_b \hat{\sigma}_{ab}\langle y_b\rangle_L+\sigma_a \tag{C8a}$$

or, in an obvious vector notation,

$$\partial_L|\bar{y}\rangle_L=2\hat{\sigma}|y\rangle_L+|\sigma\rangle . \tag{C8b}$$

The solution of (C8), with the initial condition

$$|\bar{y}\rangle_0=0 \tag{C9}$$

is

$$|\bar{y}\rangle_L=\tfrac{1}{2}(e^{2L\hat{\sigma}}-1)|w\rangle , \tag{C10}$$

where

$$|w\rangle=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} . \tag{C11}$$

Multiplying both sides of (C10) by $\langle w|$ and noting that

$$\langle w|y\rangle=\rho_1 , \tag{C12}$$

we have

$$\bar{\rho}_1=\tfrac{1}{2}\langle w|e^{2L\hat{\sigma}}-1|w\rangle . \tag{C13}$$

Equation (C13) represents the *exact* solution of our evolution equation for the centroid of the λ spectrum, for an *arbitrary* matrix of inverse MFP's $\hat{\sigma}$. Since $\hat{\sigma}$ is a real symmetric matrix, we can introduce its eigenvalues and eigenvectors by the relation

$$\hat{\sigma}|i\rangle=\hat{l}_i^{-1}|i\rangle , \tag{C14}$$

and express $\hat{\rho}_1$ of (C13) in terms of them as

$$\bar{\rho}_1=\tfrac{1}{2}\sum_{i=1}^{N}\langle w|i\rangle^2(e^{2L/\hat{l}_i}-1) . \tag{C15}$$

For large L, the largest eigenvalues of $\hat{\sigma}$, i.e., the shortest eigen-MFP $\hat{l}_1$, gives the most important contribution, thus giving the exponential behavior

$$\bar{\rho}_1=\tfrac{1}{2}\langle w|1\rangle^2(e^{2L/\hat{l}_1}-1) . \tag{C16}$$

An important particular case is that of *equivalent channels* (EC), in which the various MFP's l_a defined in Eq. (3.14c) are independent of a, i.e.,

$$l_a=l . \tag{C17}$$

In this case, the vector $|w\rangle$ of Eq. (C11) is an eigenvector of the matrix $\hat{\sigma}$, with the eigenvalue $1/l$, i.e.,

$$\hat{\sigma}|w\rangle=l^{-1}|w\rangle , \tag{C18}$$

and Eq. (C13) reduces to

$$\bar{\rho}_1=\frac{N}{2}(e^{2L/l}-1) . \tag{C19}$$

Equation (C19), valid in the NIM with EC, coincides with the result found in Ref. 8 for the IM.

APPENDIX D: THE EQUALITY OF THE AVERAGE OF FUNCTIONS OF TRACES IN THE NIM WITH EC AND THE IM

Before considering a general function of the traces (C5), we shall find it instructive to further illustrate our statement in a few simple cases. We outline the main ideas, leaving the algebraic details for the appendices that follow.

We first express the above case ρ_1 studied in Appendix C in a language that will be suitable for our future purposes. We go back to Eq. (C6) and sum over a, to get

$$H\rho_1=\sum_b \sigma_b(2y_b+1) . \tag{D1}$$

Notice that specific reference to the σ' matrix has disappeared in Eq. (D1). This does not mean, though, that the resulting $\langle\rho_1\rangle_L$ will not depend on σ' (as was improperly stated in Ref. 13), as is clear from the explicit solution (C13). In fact, (D1) alone is not sufficient to find $\langle\rho_1\rangle_L$; we need additional information on $\langle y_b\rangle$, which does depend on σ'.

For EC, Eqs. (C17) and (D1) give

$$H^{\mathrm{EC}}\rho_1=\frac{1}{l}(2\rho_1+N) . \tag{D2}$$

The evolution operator involved in the IM, to be called H^{IM}, is given explicitly in Eq. (E4); applying it to ρ_1, we

find the right-hand side of (D2) again, i.e.,

$$H^{\mathrm{EC}}\rho_1 = H^{\mathrm{IM}}\rho_1 , \tag{D3}$$

the explicit result containing ρ_1 only [see (D2)]; we thus conclude that

$$\langle \rho_1 \rangle_L^{\mathrm{EC}} = \langle \rho_1 \rangle_L^{\mathrm{IM}} , \tag{D4}$$

as was already mentioned right after Eq. (C19).

As an extension of the above example, Appendix E shows that H^{EC} acting on ρ_2 and ρ_1^2 gives the same result as H^{IM}, i.e.,

$$H^{\mathrm{EC}}\rho_2 = H^{\mathrm{IM}}\rho_2 , \tag{D5a}$$

$$H^{\mathrm{EC}}\rho_1^2 = H^{\mathrm{IM}}\rho_1^2 , \tag{D5b}$$

the explicit result containing a linear combination of ρ_1,ρ_2,ρ_1^2 only; we thus conclude that

$$\langle \rho_2 \rangle_L^{\mathrm{EC}} = \langle \rho_2 \rangle_L^{\mathrm{IM}} , \tag{D6a}$$

$$\langle \rho_1^2 \rangle_L^{\mathrm{EC}} = \langle \rho_1^2 \rangle_L^{\mathrm{IM}} , \tag{D6b}$$

in agreement with out statement. The more general definition of EC, i.e.,

$$\sigma_{ab} = \frac{1+\delta_{ab}}{(N+1)l} \tag{D7}$$

has been used. Equation (D7) reduces to (C17) upon summing over b.

Consider now a function of ρ_1. Equation (F2) shows that

$$H^{\mathrm{EC}}F(\rho_1) = H^{\mathrm{IM}}F(\rho_1) = \frac{1}{l}\left[\frac{dF}{d\rho_1}(2\rho_1+N) + \frac{2}{N+1}\frac{d^2F}{d\rho_1^2}(\rho_1+\rho_2)\right] . \tag{D8}$$

We can now draw an interesting conclusion from Eq. (D8) in the case $N=1$. In that case, ρ_2,ρ_1 are related: indeed, in the notation of Eq. (2.8),

$$\rho_1 = \lambda, \quad \rho_2 = \lambda^2 , \tag{D9}$$

so that (D8) has the structure

$$HF(\lambda) = G(\lambda) , \tag{D10a}$$

where H designates either H^{EC} or H^{IM}, implying the evolution equation

$$\partial_L \langle F(\lambda) \rangle_L = \langle G(\lambda) \rangle_L \tag{D10b}$$

in both the NIM with EC and the IM. We use the particular $F(\lambda)$

$$F(\lambda) = \delta(\lambda - \lambda') , \tag{D11}$$

and write $\langle F(\lambda) \rangle_L$ in terms of the λ-probability density $p_L(\lambda)$ as

$$\langle F(\lambda) \rangle_L = \int_0^\infty p_L(\lambda) F(\lambda) d\lambda . \tag{D12}$$

Equation (D10b) then gives the evolution equation

$$\frac{\partial}{\partial L} p_L(\lambda) = \frac{\partial}{\partial \lambda}\left[\lambda(1+\lambda)\frac{\partial p_L(\lambda)}{\partial L}\right] , \tag{D13}$$

in the NIM with EC and the IM. It then follows that

$$p_L^{\mathrm{EC}}(\lambda) = p_L^{\mathrm{IM}}(\lambda) \tag{D14a}$$

and, for arbitrary $F(\lambda)$,

$$\langle F(\lambda) \rangle^{\mathrm{EC}} = \langle F(\lambda) \rangle^{\mathrm{IM}} . \tag{D14b}$$

Equation (D13), the particular case $N=1$ of the IM result of Ref. 8 [our Eq. (5.2)], had been originally obtained by Melnikov without making the isotropy assumption. Our result is thus consistent with this fact.

This last result is extended in Appendix F to an arbitrary function $F(\rho_1,\rho_2)$ for $N=2$. The argument for arbitrary N is as follows. First, Appendix G shows that H acting on an arbitrary function $F(\rho_1,\ldots,\rho_N)$ has the structure

$$HF(\rho_1,\ldots,\rho_N) = \sum_{i=1}^{N} \frac{\partial F}{\partial \rho_i} H\rho_i + \tfrac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 F}{\partial \rho_i \partial \rho_j}[H(\rho_i\rho_j) - \rho_i H\rho_j - (H\rho_i)\rho_j] , \tag{D15}$$

in the general NIM with NEC and in the IM. The building pieces are thus $\rho_i,\rho_i\rho_j$ acted upon by H: the result of this action is the same in the NIM with EC and the IM, as shown in Appendix H; Eqs. (D3) and (D5) above are particular cases of this result. In the expression for $H(\rho_i\rho_j)$, Eq. (H2), there occurs ρ_{i+j}, and $i+j$ may exceed N; however, every $\rho_k, k>N$, can be expressed uniquely in terms of $\rho_1,\ldots,\rho_N$. Thus the result has the structure

$$HF(\rho_1,\ldots,\rho_N) = G(\rho_1,\ldots,\rho_N) , \tag{D16a}$$

$$\partial_L \langle F(\rho_1,\ldots,\rho_N) \rangle_L = \langle G(\rho_1,\ldots,\rho_N) \rangle_L , \tag{D16b}$$

in the NIM with EC and the IM. The choice $F(\rho_1,\ldots,\rho_N) = \delta(\rho_1-\rho_1')\cdots\delta(\rho_N-\rho_N')$ gives the same evolution equation

$$\partial_L p_L(\rho_1,\ldots,\rho_N) = K(\rho_1,\ldots,\rho_N) \tag{D17}$$

in the two models. The resulting p_L is thus the same,

$$p_L^{\mathrm{EC}}(\rho_1,\ldots,\rho_N) = p_L^{\mathrm{IM}}(\rho_1,\ldots,\rho_N) , \tag{D18}$$

and so is the expectation value of an arbitrary F:

$$\langle F(\rho_1,..,\rho_N) \rangle_L^{\mathrm{EC}} = \langle F(\rho_1,\ldots,\rho_N) \rangle_L^{\mathrm{IM}} . \tag{D19}$$

The N parameters $\lambda_1,\ldots,\lambda_N$ of Eq. (2.8) can be expressed uniquely in terms of the N traces $\rho_1,\ldots,\rho_N$, up to a reordering. The equality (D18) thus implies the equality

$$w_L^{\mathrm{EC}}(\lambda_1,\ldots,\lambda_N) = w_L^{\mathrm{IM}}(\lambda_1,\ldots,\lambda_N) \tag{D20}$$

for the joint probability density of $\lambda_1,\ldots,\lambda_N$ studied in Ref. 8.

APPENDIX E: PROOF OF EQS. (D5)

Applying to ρ_2 the evolution operator H of Eq. (3.21) gives

$$H\rho_2=2\sum_b \sigma_b[(yy^\dagger)_{bb}+(zz^\dagger)_{bb}] +2\sum_{ab}\sigma_{ab}(x_{bb}y_{aa}+x_{ab}y_{ba})-2\sum_a \sigma_{aa}x_{aa}y_{aa} , \tag{E1}$$

where

$$z=\alpha\beta^T , \tag{E2}$$

and x,y are defined in (C1) and (C2).

For EC we use the definition (D7). No assumption is needed for σ'_{ab}, because σ'_{ab} does not appear in Eq. (E1). We find

$$H^{\mathrm{EC}}\rho_2=\frac{2}{(N+1)l}[(2N+3)\rho_2+2(N+1)\rho_1+\rho_1^2] . \tag{E3}$$

To study the IM, we write the evolution operator H^{IM} acting on a function $F(\lambda_1,\ldots,\lambda_N)$ as in Ref. 8, i.e.,

$$H^{\mathrm{IM}}F(\lambda_1,\ldots,\lambda_N)=\frac{2}{(N+1)l}\left\{\sum_{a=1}^{N}\left[\lambda_a(1+\lambda_a)\frac{\partial^2F}{\partial\lambda_a^2}+(1+2\lambda_a)\frac{\partial F}{\partial\lambda_a}\right] +\tfrac{1}{2}\sum_{a\neq b}\frac{1}{\lambda_a-\lambda_b}\left[\lambda_a(1+\lambda_a)\frac{\partial F}{\partial\lambda_a}-\lambda_b(1+\lambda_b)\frac{\partial F}{\partial\lambda_b}\right]\right\} . \tag{E4}$$

Applying (E4) to

$$F=\rho_2=\sum_{a=1}^{N}\lambda_a^2 , \tag{E5}$$

we find again the result (E3). This proves Eq. (D5a).

Applying the H of (3.21) to ρ_1^2 we find

$$H\rho_1^2=2\rho_1\sum_b\sigma_b(2y_b+1)+2\sum_{ab}\sigma_{ab}[(\alpha\beta^T)_{ba}(\alpha^*\beta^\dagger)_{ba}+(\alpha\beta^T)_{ab}(\alpha^*\beta^\dagger)_{ba}]-2\sum_a\sigma_{aa}(\alpha\beta^T)_{aa}(\alpha^*\beta^\dagger)_{aa} . \tag{E6}$$

For EC, Eq. (D7), we find

$$H^{\mathrm{EC}}\rho_1^2=\frac{2}{(N+1)l}[(N^2+N+2)\rho_1+2(N+1)\rho_1^2+2\rho_2] . \tag{E7}$$

Applying (E4) to

$$F=\rho_1^2=\left[\sum_a\lambda_a\right]^2 , \tag{E8}$$

we find (E6) again, thus proving Eq. (D5b).

APPENDIX F: PROOF OF EQ. (D8)

Applying the evolution operator H of (3.21) to $F(\rho_1,\rho_2)$, we find

$$\begin{aligned}HF(\rho_1,\rho_2)=&\frac{\partial F}{\partial\rho_1}\sum_b(x_{bb}+y_{bb})+2\frac{\partial F}{\partial\rho_2}\left\{\sum_b\sigma_b[(y^2)_{bb}+(zz^\dagger)_{bb}]+\sum_{ab}\sigma_{ab}(x_{bb}y_{aa}+x_{ab}y_{ba})-\sum_a\sigma_{aa}x_{aa}y_{aa}\right\}\\ &+\frac{\partial^2F}{\partial\rho_1^2}\left[\sum_{ab}\sigma_{ab}(z_{ba}z_{ab}^\dagger+z_{ab}z_{ba}^*)-\sum_a\sigma_{aa}z_{aa}z_{aa}^*\right]\\ &+4\frac{\partial^2F}{\partial\rho_2^2}\left\{\sum_{ab}\sigma_{ab}[(zy^*)_{ba}(y^*z^\dagger)_{ab}+(xy^*)_{ab}(z^*y)_{ba}]-\sum_a\sigma_{aa}[(zy^*)_{aa}(z^*y)_{aa}]\right\}\\ &+2\frac{\partial^2F}{\partial\rho_1\rho_2}\left\{\sum_{ab}\sigma_{ab}[(zy^*)_{ba}z_{ab}^\dagger+z_{ba}(y^*z^\dagger)_{ab}+(zy^*)_{ab}z_{ba}^*+z_{ab}(z^*y)_{ba}]\right.\\ &\left.-\sum_a\sigma_{aa}[(zy^*)_{aa}z_{aa}^*+z_{aa}(z^*y)_{aa}]\right\} .\end{aligned} \tag{F1}$$

For EC, Eq. (D7), (F1) reduces to

$$H^{\mathrm{EC}}F(\rho_1,\rho_2)=\frac{1}{(N+1)l}\left\{(N+1)\frac{\partial F}{\partial\rho_1}(2\rho_1+N)+2\frac{\partial F}{\partial\rho_2}[(2N+3)\rho_2+2(N+1)\rho_1+\rho_1^2]\right.$$
$$\left.+2\frac{\partial^2 F}{\partial\rho_1^2}(\rho_1+\rho_2)+8\frac{\partial^2 F}{\partial\rho_2^2}(\rho_3+\rho_4)+8\frac{\partial^2 F}{\partial\rho_1\partial\rho_2}(\rho_2+\rho_3)\right\}. \tag{F2}$$

For a function of ρ_1 only we find the result of the last member in Eq. (D8). Similarly, we obtain exactly the same result (F2) if we apply H^{IM} of Eq. (E4) to $F(\rho_1,\rho_2)$, thus proving (D8).

For $N=2$, ρ_3,ρ_4 can be expressed in terms of ρ_1,ρ_2 as

$$\rho_3=-\tfrac{1}{2}\rho_1^3+\tfrac{3}{2}\rho_1\rho_2\ , \tag{F3}$$

$$\rho_4=-\tfrac{1}{2}\rho_1^4+\rho_1^2\rho_2+\tfrac{1}{2}\rho_2^2\ , \tag{F4}$$

so that (F2) contains ρ_1,ρ_2 only. Using the particular F,

$$F(\rho_1,\rho_2)=\delta(\rho_1-\rho_1')\delta(\rho_2-\rho_2')\ , \tag{F5}$$

we obtain, for the evolution of the joint probability density $p_L(\rho_1,\rho_2)$, the differential equation

$$\partial_L p_L(\rho_1,\rho_2)=-2\frac{\partial}{\partial\rho_1}[(1+\rho_1)p_L(\rho_1,\rho_2)]-\frac{\partial}{\partial\rho_2}[(7\rho_2+6\rho_1+\rho_1^2)p_L(\rho_1,\rho_2)]+\frac{\partial^2}{\partial\rho_1^2}[(\rho_1+\rho_2)p_L(\rho_1,\rho_2)]$$
$$+\frac{\partial^2}{\partial\rho_2^2}[(-\tfrac{1}{2}\rho_1^3+\tfrac{3}{2}\rho_1\rho_2-\tfrac{1}{2}\rho_1^4+\rho_1^2\rho_2+\tfrac{1}{2}\rho_2^2)p_L(\rho_1,\rho_2)]+\frac{\partial^2}{\partial\rho_1\partial\rho_2}[(-\tfrac{1}{2}\rho_1^3+\tfrac{3}{2}\rho_1\rho_2+\rho_2)p_L(\rho_1,\rho_2)] \tag{F6}$$

for the NIM with EC and the IM as well; $p_L(\rho_1,\rho_2)$ and any expectation value are thus the same in the two models. This proves the statement made right above Eq. (D15) for $F(\rho_1,\rho_2)$.

APPENDIX G: PROOF OF EQ. (D15)

Consider a function $F(\rho_1,\ldots,\rho_N)$. Let x,y be two parameters upon which the ρ_i's depend. We then find

$$\frac{\partial^2 F}{\partial x\,\partial y}=\sum_i\frac{\partial F}{\partial\rho_i}\frac{\partial^2\rho_i}{\partial x\,\partial y}+\sum_{ij}\frac{\partial^2 F}{\partial\rho_i\partial\rho_j}\frac{\partial\rho_i}{\partial x}\frac{\partial\rho_j}{\partial y}\ . \tag{G1}$$

Since

$$\frac{\partial^2\rho_i\rho_j}{\partial x\,\partial y}=\rho_i\frac{\partial^2\rho_j}{\partial x\,\partial y}+\frac{\partial\rho_i}{\partial x}\frac{\partial\rho_j}{\partial y}+\frac{\partial\rho_i}{\partial y}\frac{\partial\rho_j}{\partial x}+\frac{\partial^2\rho_i}{\partial x\,\partial y}\rho_j\ , \tag{G2}$$

we can write (G1) as

$$\frac{\partial^2 F}{\partial x\,\partial y}=\sum_i\frac{\partial F}{\partial\rho_i}\frac{\partial^2\rho_i}{\partial x\,\partial y}+\tfrac{1}{2}\sum_{ij}\frac{\partial^2 F}{\partial\rho_i\partial\rho_j}\left[\frac{\partial^2(\rho_i\rho_j)}{\partial x\,\partial y}-\rho_i\frac{\partial^2\rho_j}{\partial x\,\partial y}-\frac{\partial^2\rho_i}{\partial x\,\partial y}\rho_j\right]. \tag{G3}$$

Every one of the second derivatives occurring in Eq. (3.21) can be written as in Eq. (G3). We thus obtain (D15) for the NIM with NEC. From the structure (E4) of H^{IM} we can prove Eq. (D15) for the IM as well.

APPENDIX H: THE EFFECT OF H ON ρ_k AND $\rho_k\rho_l$

From (3.21) we can calculate $H\rho_k$ and then specialize for EC, Eq. (D7). The same result is obtained from the evolution operator (E4). We merely quote the result here:

$$H\rho_k=\frac{k}{(N+1)l}\{(N+1)\rho_k+[(\rho_1+\rho_0)\rho_{k-1}+(\rho_2+\rho_1)\rho_{k-2}+\cdots+(\rho_{k-1}+\rho_{k-2})\rho_1+(\rho_k+\rho_{k-1})\rho_0]+k(\rho_k+\rho_{k-1})\}\ . \tag{H1}$$

For $H(\rho_k\rho_l)$ also we obtain the same result from (3.21) and EC as from (E4). Explicitly, it is given by

$$H(\rho_k\rho_l)=(H\rho_k)\rho_l+\rho_k(H\rho_l)+\frac{4k^2}{(N+1)l}(\rho_{k+l}+\rho_{k+l-1})\ . \tag{H2}$$

[1]R. Landauer, Philos. Mag. **21**, 863 (1970).
[2]P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980); P. W. Anderson, *ibid.* **23**, 4828 (1981).
[3]M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. B **31**, 6207 (1985).
[4]B. Shapiro, Phys. Rev. B **35**, 8256 (1987).
[5]M. Büttiker, IBM J. Res. Develop. **32**, 317 (1988); A. D. Stone and A. Szafer, *ibid.* **32**, 384 (1988).
[6]K. A. Muttalib, J.-L. Pichard, and A. D. Stone, Phys. Rev. Lett. **59**, 2475 (1987).
[7]J.-L. Pichard, N. Zanon, Y. Imry, and A. D. Stone, J. Phys. (Paris) **51**, 587 (1990).
[8]P. A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (N.Y.) **181**, 290 (1988).
[9]P. A. Mello and B. Shapiro, Phys. Rev. B **37**, 5860 (1988).
[10]P. A. Mello, Phys. Rev. Lett. **60**, 1089 (1988).
[11]P. A. Mello, E. Akkermans, and B. Shapiro, Phys. Rev. Lett. **61**, 459 (1988).
[12]P. A. Mello and A. D. Stone, Phys. Rev. B **44**, 3559 (1991).
[13]P. A. Mello and S. Tomsovic, Phys. Rev. Lett. **67**, 342 (1991).
[14]A. D. Stone, P. A. Mello, K. A. Muttalib, and J.-L. Pichard, in *Mesoscopic Phenomena in Solids,* edited by B. L. Altshuler, P. A. Lee, and R. A. Webb, Modern Problems in Condensed Matter Sciences Series Vol. 38 (North-Holland, Amsterdam, 1991).
[15]A. D. Stone, Phys. Rev. Lett. **54**, 2692 (1985); P. A. Lee and A. D. Stone, *ibid.* **55**, 1622 (1985); P. A. Lee, A. D. Stone, and H. Fukuyama, Phys. Rev. B **35**, 1039 (1987); S. Feng, C. Kane, P. A. Lee, and A. D. Stone, Phys. Rev. Lett. **61**, 834 (1988).
[16]B. L. Al'tshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)]; B. L. Al'tshuler and D. E. Khmel'nitskii, *ibid.* **42**, 291 (1985) [*ibid.* **42**, 359 (1986)]; B. L. Al'tshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
[17]C. P. Umbach, S. Washburn, R. B. Laibowitz, and R. A. Webb, Phys. Rev. B **30**, 4048 (1984); R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, Phys. Rev. Lett. **54**, 2696 (1985); S. Washburn, C. P. Umbach, R. B. Laibowitz, and R. A. Webb, Phys. Rev. B **32**, 4789 (1985).
[18]M. Hamermesh, *Group Theory and Its Applications to Physical Problems* (Addison-Wesley, Reading, MA, 1962).
[19]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon & Breach, New York, 1963), Vol. 1, Chap. 4.
[20]P. A. Mello and J.-L. Pichard, J. Phys. (Paris) I **1**, (1991) 493.
[21]Since evanescent channels are neglected, it would be more appropriate to say that the present approach applies to a set of N-interacting chains, described by N-coupled 1D Schrödinger equations.
[22]P. Pereyra (private communication).
[23]A model where the individual scatterers, or thin slices, are described by a δ potential in the x direction, with random channel matrix elements, is currently under study. Preliminary results in 1D coincide with those of the present model when $k_F l \gg 1$.
[24]E. Akkermans, P. E. Wolf, R. Maynard, and G. Maret, J. Phys. (Paris) **49**, 77 (1988).
[25]P. E. Wolf, G. Maret, E. Akkermans, and R. Maynard, J. Phys. (Paris) **49**, 63 (1988).
[26]J.-L. Pichard (private communication).
[27]P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[28]S. Godoy and P. A. Mello, Phys. Rev. B **46**, 2346 (1992).
[29]W. J. Skocpol, P. M. Mankiewick, R. E. Howard, L. D. Jackel, D. M. Tennant, and A. D. Stone, Phys. Rev. Lett. **58**, 2347 (1987); C. P. Umbach, S. Santhanam, C. Van Hsendouck, and R. A. Webb, Appl. Phys. Lett. **50**, 1289 (1987).
[30]H. U. Baranger, A. D. Stone, and D. P. Di Vincenzo, Phys. Rev. B **37**, 6521 (1988).
[31]D. Kowal, U. Sivan, O. Entin-Wohlman, and Y. Imry, Phys. Rev. B **42**, 9009 (1990).
[32]V. Gopar, M. Martínez, and P. A. Mello (unpublished).
[33]P. A. Mello (unpublished).